AD 296 787

AFESD - TDR - 62- 273

284

AFESD - TDR - 62- 273

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

OPTIMUM INCORPORATION OF EXTERNAL OBSERVATIONS WITH A MARINE INERTIAL NAVIGATION SYSTEM

D. J. SAKRISON

Group 22

TECHNICAL REPORT NO. 289

30 NOVEMBER 1962

LEXINGTON MASSACHUSETTS

ABSTRACT

When a marine vessel is used as a platform for tracking space vehicles, the accuracy of the tracking is critically limited by errors in the ship's estimated position and orientation. These reference data are usually estimated by combining the output of an inertial navigation system with additional external position, velocity and/or orientation measurements. To meet stringent accuracy requirements for this estimation, optimum statistical computation procedures for combining the data are required.

This report develops a discrete-time, linear model for the error propagation in an inertial navigation system. This model makes it possible to derive an optimum recursive data processing procedure for combining the inertial system output with the external fixes. A recursive formula for the minimum-mean-square error in estimated position, velocity and orientation is also obtained. The various formulas are directly applicable to broad classes of inertial navigation systems and external measurements. A general discussion is provided on the computational problems associated with both actual data processing and the performance of system-error analyses.

TABLE OF CONTENTS

OPTIMUM INCORPORATION OF EXTERNAL OBSERVATIONS WITH A MARINE INERTIAL NAVIGATION SYSTEM

I. INTRODUCTION

The problem that motivated the analysis contained in this report is connected with the use of marine vessels as platforms for devices used to track missiles, satellites and interplanetary probes. One of the major difficulties associated with these marine-stationed tracking platforms is the degradation of the usefulness of the tracking data caused by uncertainties as to the ship's latitude, longitude and orientation with respect to north and vertical. Techniques that can be used to determine these reference data for a marine vessel can be subdivided into two basic classes. In the first class is the use of direct measurements such as star fixes, tracking of satellites, shore-based electromagnetic navigation aids and underwater sonar beacons. In the second class, consisting of direct navigational procedures, the technique of prime interest is the inertial navigation system.† These two methods, external measurements and inertial navigation systems, have different performance characteristics. An inertial navigation system is capable of high accuracy for short periods of time, but the errors can build up to an unacceptable level. External measurements provide information only at the time of measurement, and any single measurement might not be accurate enough to meet requirements or might not give enough information to completely specify the desired aspects of the vessel's position, velocity and orientation. Because of economic as well as physical limitations, the required accuracies are not always achievable by merely improving the inertial navigator or the basic measurement systems.

When combined in an efficient manner, however, the two techniques complement each other and can possibly provide a resulting capability far superior to either single procedure. The inertial navigator allows the use of incomplete external measurements taken at different times, and statistical smoothing of the random errors in the external measurements can be achieved. In turn, the external measurements help control the error build-up inherent in the inertial navigation system. Since extensive computation facilities for data processing are usually available to the marine-stationed tracking platform, one can use optimum data processing techniques for combining external measurements with the data obtained from an inertial navigation system to provide a best estimate of a ship's position and orientation. Knowledge of such optimum computation procedures also provides answers to such related operational questions as: What external measurements are to be taken, at what time, and in what combinations to provide the most effective results? When can inexpensive external measuring devices be combined with an inexpensive inertial navigation system to provide performance characteristics competitive with far more

†The inertial navigation system could be considered as using external measurements in that it relies on gravitational forces. However, the dichotomy is useful for the sake of exposition.

expensive inertial navigation systems? If underwater sonar beacons are being employed, how should they be placed and in what pattern about them should the marine vessel sail?

This report develops a theory and derives explicit formulas that provide answers to the above questions. The results are general in nature and thus are applicable to problems beyond those that were of interest initially. A discrete-time model for the error propagation of a marine inertial navigation system is developed and, on the basis of this model, an optimum method for combining the external measurements with the inertial navigation system is derived. The optimum technique provides a reasonable and workable method of data processing, but even if restrictions prevent its explicit use, the theory provides a valuable standard of comparison, as well as a good starting point for modifications. The derivation of the data processing procedure also includes derivation of formulas that determine how the various errors in the inertial navigation system and external measurements are propagated through the optimum data processing technique. These formulas thus provide a very powerful tool for comparing and evaluating the effectiveness of different external measurement and inertial navigation system configurations.

Explicit formulas are provided, but, because of the inherent complexity of the error behavior of inertial navigation systems, closed-form solutions are usually not possible. The formulas, however, are directly solvable on a digital computer (some analog equipment could also be used if desired) by the use of straightforward techniques. For example, the formula governing the error propagation in time is a nonlinear recursive equation whose solution can be calculated directly in time from known initial conditions. Since the necessary computer programs have not been written, no explicit numerical results are included in this report.

The material in this report assumes familiarity with inertial navigation systems; therefore, the treatment of the systems themselves, although mathematically complete, is very abbreviated. The development of the theory for the optimum techniques is more expository, but assumes a basic knowledge of random variables and stochastic processes.

II. GENERAL DISCUSSION

Because of the technical nature of the theory and formulas to be developed, it is appropriate to preface the actual derivations with a general discussion of the basic assumptions and methods of analysis to be used. The remainder of the report (Secs. III, IV and V) will be of interest primarily to those readers who wish to employ the general theory or use the explicit formulas to solve particular problems.

The model of the inertial navigation system to be considered includes:

(a) The possible incorporation of a velocity log for measuring the vessel's velocity and the effect of errors in these measurements.

(b) The effect of errors due to gyro drift and accelerometer bias.

(c) Incorporation of an extremely broad class of external measurements.

(d) The propagation of all seven components of error, two location coordinates and their associated velocities and three attitude angles. (The term "position estimation," as used throughout the remainder of this report, actually refers to all seven quantities, not just the location of the ship on the surface of the earth.)

Section III develops a discrete-time model of the error propagation in the inertial navigator. In deriving the error model, it is assumed that the other accelerations acting on the platform are small compared with the acceleration of the earth's gravitational field and that the velocity of the ship with respect to the earth's surface is small compared with the velocity of the earth's surface with respect to some inertial reference. We assume that the velocity-log errors and accelerometer-bias errors (additive errors in the accelerometer output) can be represented by continuous, stationary, zero mean, Markov random processes. Further, it is assumed that the values of the associated correlation functions for time shifts larger than the time between fix observations will be very small compared with the values for zero time shift. Stationary first-order Markov processes are assumed for the gyro drifts, but here it is not assumed that the correlation times are small with respect to the time between fix observations. The linearized model for the error propagation is of the form

$$\underline{x}(t + T) = \Phi(t + T)\,\underline{x}(t) + \underline{w}(t) \quad , \tag{1}$$

in which T is the time between external measurements. The column vector $\underline{x}(t)$ is a ten-dimensional vector, the first seven entries of which are the seven error variables while the last three are variables that are the discrete-time equivalent of the gyro-drift inputs (equivalent in the sense that they produce the same response at all sample times as the original continuous gyro-drift inputs). The column vector $\underline{w}(t)$ is also a ten-dimensional vector, the first seven entries of which are inputs that directly represent the effects of accelerometer bias and velocity-log errors. The last three entries of $\underline{w}(t)$ are inputs that, when passed through a first-order system, result in the equivalent gyro-drift inputs (the last three entries of $\underline{x}$). The 10×10 matrix Φ thus includes both the transition matrix of the error-propagation system and the transition matrix of the 1^{st}-order system used to generate the three equivalent gyro-drift inputs. The input $\underline{w}(t)$ is white in the sense that†

$$E\{\underline{w}_t \underline{w}_{t+kT}^T\} = 0 \qquad k = \text{integer};\ k \neq 0 \quad ,$$

where the superscript T denotes transposition. The matrix Φ is a constant for picket ship operation (operation confined to a distance of several miles from a fixed point) and varies with location for normal marine operation.

Section IV treats the estimation problem. For the external measurements, we assume a rather general form

$$\underline{r}(t) = \underline{F}[\underline{q}(t)] - \underline{v}(t) \quad ,$$

in which the vector $\underline{r}$ is the observation taken, $\underline{q}(t)$ is the seven-vector giving the true quantities associated with the inertial navigator operation, and $\underline{v}(t)$ is the measurement error. This error is assumed to have zero mean and to be white in the sense that

$$E\{\underline{v}_t \underline{v}_{t+kT}^T\} = 0 \qquad k = \text{integer};\ k \neq 0 \quad .$$

The output of the inertial navigator is $\underline{y}(t) = \underline{q}(t) + \underline{x}(t)$. The variables used by our over-all system to estimate $\underline{x}_t$ are

$$\begin{aligned} \underline{z}(\tau) &= \underline{F}[\underline{y}(\tau)] - \underline{r}(\tau) \\ &\doteq H(\tau)\underline{x}(\tau) + \underline{v}(\tau) \qquad \tau = t - kT,\ k = 0, 1, 2, \ldots \quad , \end{aligned} \tag{2}$$

†The symbol $\underline{x}(t)$ denotes a sample function of a random process (a function of time that is the result of a single realization, or experiment) while the symbol $\underline{x}_t$ denotes a random variable (an observation made at a single fixed time that is a function of the realization, or experiment observed). Thus, a time average is defined only for $\underline{x}(t)$ and a statistical average only for $\underline{x}_t$.

in which we have expanded $\underline{F}[\underline{y}(\tau)]$ in a Taylor's series about $\underline{y}(\tau) = \underline{q}(\tau)$; that is,

$$H_{ij}(\tau) = \left. \frac{\partial F_i[s(\tau)]}{\partial s_j(\tau)} \right|_{\underline{y}(\tau) = \underline{q}(\tau)} \quad .$$

Thus, the external measurements are also linearized.

The use of a linearized model such as that represented by Eqs. (1) and (2) is justified in most problems of interest, since we are primarily concerned with highly accurate systems having small errors. Of course, in any particular application, one must always determine whether or not the linearized model is a valid representation of the important aspects of the physical situation.

The estimation scheme developed in Sec. IV is linear; that is, the optimum estimate is expressed as

$$\hat{\underline{x}}(t) = \sum C_k \underline{z}(t-k) \quad ,$$

in which the sum extends over all measurements. The reasons for using a linear estimation scheme are twofold:

(a) The solution to the linear estimation problem is tractable.

(b) It is unrealistic to assume that we would have good estimates of more than second-order statistics; using only second-order statistics, the linear estimate is optimum.† Moreover, it seems reasonable to assume that the sources generating navigator errors would be Gaussian. Since the error-propagation system is linear to a good approximation, the processes that we are trying to estimate may be assumed Gaussian to a good approximation. Under these conditions, the optimum mean-square estimator is a linear estimator.

The solution to the estimation problem is similar to Kalman's recursive formulation.[2] The results are:

(a) A recursive formula that provides the optimum data processing procedure.

(b) A recursive relation for determining the covariance matrix of the error in estimating the position of the marine vessel.

The recursive formulations given in Sec. IV are nonlinear and, for almost all situations of interest, require machine solution. Sec. V discusses the computational steps necessary to carry out a machine solution.

One final point: the formulas to follow are complex because of the generality of the assumed model. In any explicit application, extensive simplifications may be possible. Similarly, the analysis itself can be extended in various directions by using the same basic techniques employed in this report. To cite just one example, the theory is directly extendible to external-fix measurement errors that are correlated in time and that have a nonzero bias component.

III. MODEL OF THE NAVIGATOR ERROR PROPAGATION

In this section we proceed directly to the problem of deriving a model for the error propagation in a marine inertial navigation system. The reader unfamiliar with inertial navigation systems may wish to consult the literature, since certain of our analyses are succinct rather than

† We consider the optimum or best estimate of position to be the one that simultaneously minimizes the variance of each component of position. It is shown in Sec. IV that this estimate also yields an ellipsoid of concentration[1] that falls inside the ellipsoid of concentration of any other estimate.

expository. There is at least one textbook which covers the general subject,[3] although the author found an unpublished set of notes[4] to be somewhat more informative. The idea behind an inertial navigation system is quite basic. A gyro-stabilized platform is used to orient two accelerometers. These sense components of acceleration in the east and north directions, and these accelerations are integrated twice by a computer to obtain position. The computer also computes the torques necessary to keep the platform orientation north and east as the earth rotates and the ship moves.

Many marine inertial navigation systems also include velocity-log feedback in which estimates of the vessel's velocity, obtained from an electromagnetic velocity meter, are fed into the system to provide damping of certain errors. Although such velocity information could be considered external measurements, the analyses to follow include velocity-log feedback as an intrinsic part of the inertial navigation system. It will be seen, however, that this approach greatly complicates the equations. The possibilities of treating the velocity-log data as external measurements are briefly discussed in Sec. V.

A. Definitions of Symbols and Coordinate Systems

Throughout this report, vector quantities are indicated by underlining. The conventional latitude-longitude system is used; that is,

λ = degrees north (+) latitude

and

Λ = degrees east (+) longitude

are the variables describing the ship's position. The operation of the navigation system will be based on a set of coordinates whose origin is at the center of the earth and whose z-axis passes through the center of the ship. These coordinates will be oriented with the x-axis pointed north and the y-axis pointed west. In the remainder of this report, this coordinate system will be referred to simply as the system coordinates. We will also need to distinguish between three such sets of system coordinates:

(1) x, y, z: The set of system coordinates based on the true position of the ship.

(2) x_c, y_c, z_c: The set of system coordinates whose orientation is based on the position of the ship calculated by the navigation system computer.

(3) x_p, y_p, z_p: The set of system coordinates aligned with the orientation of the stable platform, i.e., x_p parallel to x accelerometer.

The difference between the last two sets of coordinates is due to the curvature of the earth and the existence of error in the navigator-calculated position.

Note that these three coordinate systems are all related to one another by rotational transformations. A rotational transformation is specified by three angles. If all these angles are small and sines are approximated by the angle and cosines by one, then the rotational transformation can be (approximately) specified by a linear transformation in which these angles appear as coefficients. We assume that our error angles are small. Thus, the disparity among the three coordinate systems can be described by the vectors:

$\underline{\varphi}$ = vector error angle between the true axes and platform axes,

$\underline{\delta\Theta}$ = vector error angle between the true axes and computer axes,

$\underline{\psi}$ = vector error angle between computer axes and platform axes,

$\underline{\varphi} = \underline{\psi} + \underline{\delta\Theta}$.

Each of these vector quantities has three components; e.g., $\underline{\psi}$ has the components ψ_x, ψ_y, ψ_z, representing rotations about the x, y, and z axes, respectively.

We also introduce the following quantities:

$\underline{A}$ = acceleration sensed by the accelerometer.

$\underline{R}$ = position vector of the vehicle as measured in a coordinate system aligned with x, y and z whose origin is at the center of the earth, i.e., the system coordinates. Note that $x \equiv y \equiv 0$, $z \equiv R$, R = earth's radius = $|\underline{R}|$.

$\dot{\underline{R}}$ = velocity vector of vehicle in the system coordinates. Note that $\dot{\underline{R}} \equiv \underline{0}$.

$\underline{g}_m(\underline{R})$ = mass attraction of the earth at position $\underline{R}$.

$\underline{\omega}$ = angular velocity of system coordinates relative to an inertial set of coordinates, that is, a set of coordinates stationary with respect to the fixed stars.

$\underline{\Omega}$ = angular velocity of earth.

Ω = frequency of earth's angular rotation about the North Pole = $|\underline{\Omega}|$.

$\Delta\lambda$ = error in latitude = $\lambda_c - \lambda_{actual}$, in which λ_c is the computed latitude. Note that $\Delta x = R\Delta\lambda$, $\Delta y = -R \cos\lambda \, \Delta\Lambda$.

$\Delta\dot{\lambda}$ = derivative of $\Delta\lambda$ with respect to time.

δV_x = error in x-component of ship's velocity log [an electromagnetic (actually magnetohydrodynamic) device for measuring the ship's velocity].

K = feedback from velocity log (units of inverse seconds).

δA_x = bias error in the x-component of the accelerometer output; that is, δA_x is a random process which constitutes the additive error in the x-accelerometer output.

ϵ_x = x-component of gyro drift.

Similar definitions apply to δV_y, δA_y, ϵ_y and ϵ_z.

In the next two sections we derive the differential equations governing the error propagation in the navigator. These equations can be divided into two groups. The first group is the set of differential equations for $\Delta\lambda$ and $\Delta\Lambda$ in which ψ_x, ψ_y and ψ_z appear as driving terms. The second group of equations describes the propagation of the terms ψ_x, ψ_y and ψ_z.

B. Equations Governing the Propagation of $\Delta\lambda$ and $\Delta\Lambda$

Recalling the method of operation of the inertial navigator, we see that the fundamental equation describing the system is

$$\underline{A} = \left(\frac{d^2\underline{R}}{dt^2}\right)_I - \underline{g}_m(\underline{R}) \quad , \tag{3}$$

in which I indicates that the differentiation is carried out with respect to an inertial set of coordinates. But, using the general expression relating differentiation in two coordinate systems whose relative motion is only rotation, we obtain

$$\left(\frac{d\underline{R}}{dt}\right)_I = \dot{\underline{R}} + \underline{\omega} \times \underline{R} \quad , \tag{4}$$

in which $\underline{\omega} \times \underline{R}$ is the cross product of the vectors $\underline{\omega}$ and R. Also,

$$\left(\frac{d^2\underline{R}}{dt^2}\right)_I = \left[\frac{d}{dt}\left(\frac{d\underline{R}}{dt}\right)_I\right]_{\text{Sys Coord}} + \omega \times \left(\frac{d\underline{R}}{dt}\right)_I \quad . \tag{5}$$

Substituting Eq. (4) into Eq. (5), we obtain

$$\left(\frac{d^2\underline{R}}{dt^2}\right)_I = [\frac{d}{dt}(\dot{\underline{R}} + \underline{\omega} \times \underline{R})]_{\text{Sys Coord}} + \underline{\omega} \times (\dot{\underline{R}} + \underline{\omega} \times \underline{R})$$

$$= \ddot{\underline{R}} + 2\underline{\omega} \times \dot{\underline{R}} + \underline{\omega} \times (\underline{\omega} \times \underline{R}) + \dot{\underline{\omega}} \times \underline{R} \quad . \tag{6}$$

Substituting Eq. (6) into Eq. (3) and recalling that $\dot{\underline{R}} \equiv 0$, we obtain

$$\underline{A} = -\underline{g_m}(\underline{R}) + \underline{\omega} \times (\underline{\omega} \times \underline{R}) + \dot{\underline{\omega}} \times \underline{R} \quad . \tag{7}$$

Now,

$$x = y = 0, \quad z = R \quad .$$

Further,

$$\omega_x = (\Omega + \dot{\Lambda}) \cos\lambda, \quad \omega_y = \dot{\lambda}, \quad \omega_z = (\Omega + \dot{\Lambda}) \sin\lambda \quad .$$

Thus Eq. (7) becomes

$$A_x = R\ddot{\lambda} + R(\Omega + \dot{\Lambda})^2 \cos\lambda \sin\lambda - g_x(\overline{R})$$

$$A_y = -R\cos\lambda\ddot{\Lambda} + 2R(\Omega + \dot{\Lambda})\dot{\lambda}\sin\lambda - g_y(\overline{R}) \quad . \tag{8}$$

The computation scheme used by the computer is to calculate the terms

$$R(\Omega + \dot{\Lambda})^2 \cos\lambda \sin\lambda, \quad g_x(\overline{R}), \quad 2R(\Omega + \dot{\Lambda})\dot{\lambda}\sin\lambda, \quad g_y(\overline{R})$$

and subtract them from the output of the accelerometer, thus obtaining $R\ddot{\lambda}$ and $R\cos\lambda\ddot{\Lambda}$. These are integrated twice to calculate position. The velocity-log data is incorporated into a simple feedback loop to damp certain errors. The block diagram of the navigation system is shown in Fig. 1.

Writing out the equations for the block diagram, we have the equations which govern the error propagation:

$$R(\ddot{\lambda} + \Delta\ddot{\lambda}) = R\ddot{\lambda} + R[(\Omega + \dot{\Lambda})^2 \sin\lambda\cos\lambda - (\Omega + \dot{\Lambda} + \Delta\dot{\Lambda})\sin(\lambda + \Delta\lambda) \cdot \cos(\lambda + \Delta\lambda)]$$
$$-KR\Delta\dot{\lambda} + K\delta V_x - (\psi_y A_z - \psi_z A_y) - [g_x(\overline{R}) - g_x(R + \Delta R)] + \delta A_x \quad , \tag{9}$$

$$-R\cos(\lambda + \Delta\lambda)(\ddot{\Lambda} + \Delta\ddot{\Lambda}) = -R\ddot{\Lambda}\cos\lambda + 2R[(\Omega + \dot{\Lambda})\dot{\lambda}\sin\lambda - (\Omega + \dot{\Lambda} + \Delta\dot{\Lambda})(\dot{\lambda} + \Delta\dot{\lambda})$$
$$\times \sin(\lambda + \Delta\lambda)] + KR\cos(\lambda + \Delta\lambda)(\dot{\Lambda} + \Delta\dot{\Lambda}) - KR\cos\lambda[\dot{\Lambda} + (\delta V_y)/R]$$
$$-(\psi_z A_x - \psi_x A_z) + [g_y(\overline{R}) - g_y(\overline{R} + \overline{\Delta R}) + \delta A_y] \quad . \tag{10}$$

We assume that $\Delta\lambda \ll 1$ and $\Delta\Lambda \ll 1$; hence,

$$\sin(\lambda + \Delta\lambda) \approx \sin\lambda + \Delta\lambda\cos\lambda,$$
$$\cos(\lambda + \Delta\lambda) \approx \cos\lambda - \Delta\lambda\sin\lambda.$$

Now we assume that the magnitude of the earth's gravitational attraction at sea level is known for the region of operation. If this is so, then direction is the only unknown quantity, and

$$\underline{g}(\underline{R}) - \underline{g}(\underline{R} + \underline{\Delta R}) = -g\underline{\Delta R} \quad .$$

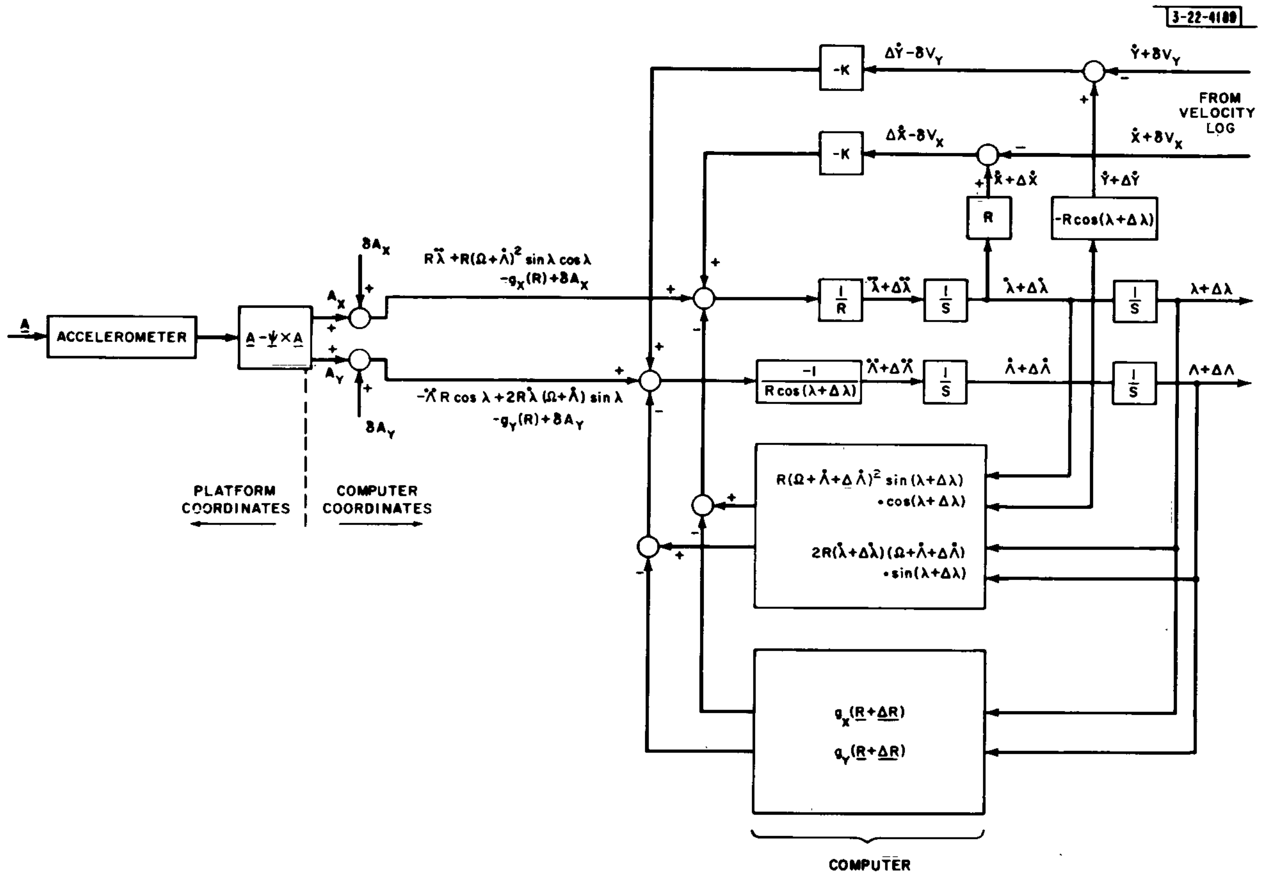


Fig. 1. Block diagram of latitude-longitude inertial navigation system.

Hence,

$$-g_x(\underline{R}) + g_x(\underline{R} + \underline{\Delta R}) = g\Delta\lambda \quad , \tag{11}$$

$$-g_y(\underline{R}) + g_y(\underline{R} + \underline{\Delta R}) = -g\Delta\Lambda \quad . \tag{12}$$

It should be noted that we have assumed a spherical earth. Although the earth's ellipticity cannot be ignored in the actual operation of the navigation system, the manner in which it affects error propagation is secondary and will be ignored. Similarly, for marine operation, we can make the further assumptions:

$$\Omega \gg \dot{\lambda} \quad , \quad \Omega \gg \dot{\Lambda} \quad ,$$

$$A_x \quad , \quad A_y \quad , \quad K\dot{\Lambda} \quad \text{and} \quad R\ddot{\Lambda} \text{ are negligible with respect to } g \quad . \tag{13}$$

We also note that

$$\Omega^2 \approx \frac{1.34}{10^{10}} \quad \text{cycles/sec} \quad , \quad \frac{g}{R} \approx \frac{8.0}{10^7} \quad \text{cycles/sec} \quad ;$$

hence,

$$\Omega^2 \ll \frac{g}{R} \quad . \tag{14}$$

We also assume that $\Delta\lambda$ and $\Delta\Lambda$ are of the same order of magnitude, and that ψ_x, ψ_y and ψ_z are all of the same magnitude. If we apply this assumption and the approximations (11) through (14) to Eqs. (9) and (10), we obtain, after dropping second-order terms,

$$\ddot{\Delta\lambda} + 2C_1\omega_2\dot{\Delta\Lambda} + K\dot{\Delta\lambda} + \omega_s^2\Delta\lambda = \frac{K}{R}\,\delta V_x + \frac{\delta A_x}{R} + \frac{g}{R}\,\psi_y \quad , \tag{15}$$

$$\ddot{\Delta\Lambda} - 2\,\frac{\omega_2}{C_1}\,\dot{\Delta\lambda} + K\dot{\Delta\Lambda} + \omega_s^2\Delta\Lambda = -\frac{K}{RC_1}\,\delta V_y - \frac{\delta A_y}{R} + \frac{g}{RC_1}\,\psi_x \quad , \tag{16}$$

in which

$$\omega_s = \text{Schuler Frequency} = \sqrt{\frac{g}{R}} \quad ,$$

$$\omega_2 = C_2\Omega \quad , \quad \omega_1 = C_1\Omega \quad ,$$

$$C_1 = \cos\lambda \quad ,$$

$$C_2 = \sin\lambda \quad .$$

If we are considering picket ship operation, then C_1 and C_2 are constant and our error propagation is determined by a set of linear differential equations with constant coefficients. For usual marine use we have linear differential equations with coefficients that vary with position.

The quantities ψ_x and ψ_y act as drives to Eqs. (15) and (16). We must next write the equations that determine the propagation of the error angle $\underline{\psi}$.

C. Equations Governing the Propagation of $\underline{\psi}$

The derivations of this section follow closely those of Kachickas.[4] To start, let us recall that we have defined the three sets of coordinate axes:

x, y, z: true position coordinate axes ,

x_p, y_p, z_p: platform coordinate axes ,

x_c, y_c, z_c: computer coordinate axes .

All three coordinate axes have their origins at the true position of the ship, but the three axes are not perfectly aligned with one another.

Let

$\underline{x}$ denote an arbitrary vector in the x, y, z system ,

$\underline{x}_p$ denote the same vector in the x_p, y_p, z_p system ,

$\underline{x}_c$ denote the same vector in the x_c, y_c, z_c system .

Then, to a first-order approximation,

$$\underline{x}_p = \Phi \underline{x} \quad \text{and} \quad \underline{x}_c = \Theta \underline{x} \quad ,$$

in which

$$\Phi = \begin{bmatrix} 1 & \varphi_z & -\varphi_y \\ -\varphi_z & 1 & \varphi_x \\ \varphi_y & -\varphi_x & 1 \end{bmatrix} \quad , \quad \Theta = \begin{bmatrix} 1 & \delta\Theta_z & -\delta\Theta_y \\ -\delta\Theta_z & 1 & \delta\Theta_x \\ \delta\Theta_y & -\delta\Theta_x & 1 \end{bmatrix} \quad .$$

We have previously defined

$$\underline{\varphi} = \begin{bmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{bmatrix} \quad , \quad \underline{\delta\Theta} = \begin{bmatrix} \delta\Theta_x \\ \delta\Theta_y \\ \delta\Theta_z \end{bmatrix} \quad .$$

Now, by definition,

$$\underline{\omega}_p = \underline{\omega} + \dot{\underline{\varphi}} \quad . \tag{17}$$

The platform velocity may also be expressed in terms of the gyro drift $\underline{\epsilon}$ and the angular velocity $\underline{\omega}_c$, which, according to the computer calculations must be applied to the platform to maintain its north, east, up orientation. In terms of the true coordinates, this angular velocity is

$$\Theta^{-1} \underline{\omega}_c + \underline{\epsilon} \quad .$$

Hence,

$$\underline{\omega}_p = \Phi[\Theta^{-1}\underline{\omega}_c + \underline{\epsilon}]$$

$$= (\Phi\Theta^{-1})\,\underline{\omega}_c + \Phi\underline{\epsilon} \quad . \tag{18}$$

By using the given expressions for Φ and Θ and noting that, to a first-order approximation,

$$\Theta^{-1} = \begin{bmatrix} 1 & -\delta\Theta_z & \delta\Theta_y \\ \delta\dot{\Theta}_z & 1 & -\delta\Theta_z \\ -\delta\Theta_y & \delta\Theta_x & 1 \end{bmatrix} \quad ,$$

we obtain

$$\underline{\omega}_p = \underline{\omega}_c + (\underline{\varphi} - \underline{\delta\Theta}) \times \underline{\omega}_c + \underline{\epsilon} + \underline{\varphi} \times \underline{\epsilon} \quad . \tag{19}$$

Now, again by definition,

$$\underline{\omega}_c = \underline{\omega} + \underline{\delta\Theta} \quad . \tag{20}$$

Substituting Eq. (20) into Eq. (19) and ignoring second-order terms, we obtain

$$\underline{\omega}_p = \underline{\omega} + \underline{\delta\dot{\Theta}} + (\underline{\varphi} - \underline{\delta\Theta}) \times \underline{\epsilon} \quad . \tag{21}$$

By equating this expression to Eq. (17) we obtain

$$\underline{\dot{\varphi}} - \underline{\delta\dot{\Theta}} + (\underline{\varphi} - \underline{\delta\Theta}) = \underline{\epsilon} \quad ,$$

or

$$\underline{\dot{\psi}} + \underline{\omega} \times \underline{\psi} = \underline{\epsilon} \quad . \tag{22}$$

Expanding Eq. (22) and substituting in the expressions for ω_x, ω_y and ω_z, given following Eq. (7), we obtain

$$\dot{\psi}_x + \dot{\lambda}\psi_z - (\Omega + \dot{\Lambda})\, C_2\psi_y = \epsilon_x \quad ,$$

$$\dot{\psi}_y + C_2(\Omega + \dot{\Lambda})\, \psi_x - C_1(\Omega + \dot{\Lambda})\, \psi_z = \epsilon_y \quad ,$$

$$\dot{\psi}_z + C_2(\Omega + \dot{\Lambda})\, \psi_y - \dot{\lambda}\psi_x = \epsilon_x \quad .$$

If we ignore $\dot{\Lambda}$ and $\dot{\lambda}$ with respect to Ω and assume that ψ_x, ψ_y and ψ_z are of the same magnitude, the above equations become

$$\dot{\psi}_x - \omega_2\psi_y = \epsilon_x \quad , \tag{23}$$

$$\dot{\psi}_y + \omega_2\psi_x - \omega_1\psi_z = \epsilon_y \quad , \tag{24}$$

$$\dot{\psi}_z + \omega_2\psi_y = \epsilon_x \quad , \tag{25}$$

in which

$$\omega_2 = C_2\Omega \quad , \quad \omega_1 = C_1\Omega \quad .$$

3-22-4180

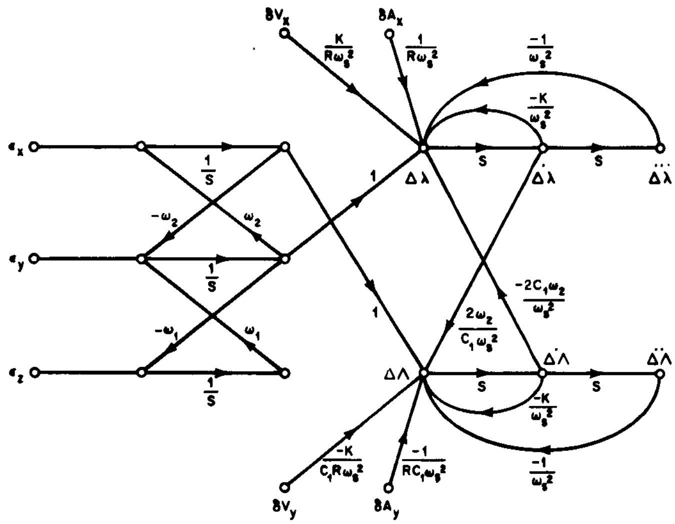
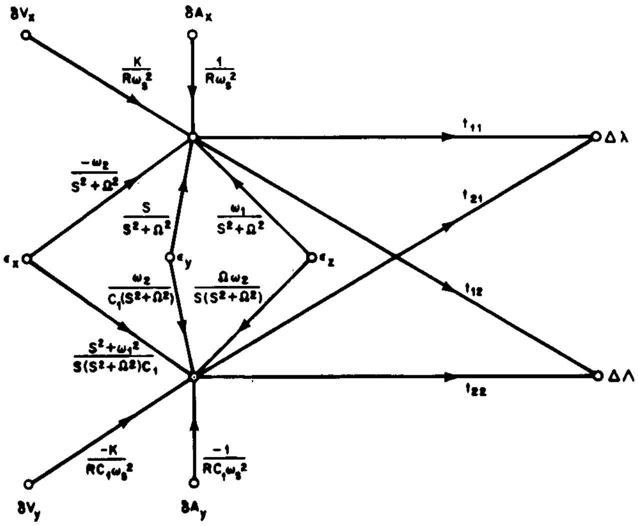


Fig. 2. Flow graph expressing the dynamics of the error propagation.

3-22-4181



$$t_{11} = \frac{(S^2+\Omega^2)^2 \omega_s^2 (S^2+KS+\omega_s^2)}{D}$$

$$t_{12} = \frac{2\omega_2 \omega_s^2 S (S^2+\Omega^2)^2}{C_1 D}$$

$$t_{21} = \frac{-2C_1 \omega_2 \omega_s^2 S (S^2+\Omega^2)^2}{D}$$

$$t_{22} = \frac{\omega_s^2 (S^2+\Omega^2)^2 (S^2+KS+\omega_s^2)}{D}$$

$$D = \left[S^2+\Omega^2\right]^2 \left[(S^2+KS+\omega_s^2)^2 - 4\omega_2^2 S^2\right]$$

Fig. 3. Reduced flow graph.

D. Complete Error Propagation Dynamics

The five equations (15), (16), (23), (24) and (25) completely describe the propagation of errors in the navigation system. Our first objective will be to obtain the transfer functions and impulse responses relating $\Delta\lambda$ and $\Delta\Lambda$ to the various inputs. This can be accomplished by first drawing the flow graph expressing the five equations:

$$\ddot{\Delta\lambda} + 2C_1\omega_2\dot{\Delta\Lambda} + K\dot{\Delta\lambda} + \omega_s\Delta\lambda = \frac{K}{R}\,\delta V_x + \frac{\delta A_x}{R} + \frac{g}{R}\,\psi_y \qquad \text{[Eq. (15)]}$$

$$\ddot{\Delta\Lambda} - 2\,\frac{\omega_2}{C_1}\,\dot{\Delta\lambda} + K\dot{\Delta\Lambda} + \omega_s\Delta\Lambda = -\frac{K}{RC_1}\,\delta V_y - \frac{\delta A_y}{R} + \frac{g}{RC_1}\,\psi_x \qquad \text{[Eq. (16)]}$$

$$\dot{\psi}_x - \omega_2\psi_y = \epsilon_x \qquad \text{[Eq. (23)]}$$

$$\dot{\psi}_y + \omega_2\psi_x - \omega_1\psi_z = \epsilon_y \qquad \text{[Eq. (24)]}$$

$$\dot{\psi}_z + \omega_2\psi_y = \epsilon_x \quad . \qquad \text{[Eq. (25)]}$$

This flow graph is shown in its original form in Fig. 2 and in reduced form in Fig. 3; the details of the reduction are omitted.

In order to proceed further, we must factor the denominator term D, which appears in the transfer functions in Fig. 3:

$$D = (S^2 + \Omega^2)^2\,[(S^2 + KS + \omega_s^2)^2 - 4\omega_2^2 S^2]$$

$$\approx (S^2 + \Omega^2)^2\,[(S^2 + KS + 1.0821\,\omega_s^2)\,(S^2 + KS + 0.9241\,\omega_s^2)] \quad . \qquad (26)$$

Now, from this we could obtain the necessary transfer functions directly. The resulting expressions, however, would be extremely unwieldy. For this reason we consider approximating D by

$$D \approx (S^2 + \Omega^2)^2\,(S^2 + KS + \omega_s^2)^2 \quad . \qquad (27)$$

In order to estimate the error involved, consider the time-response terms that would result from each expression. From Eq. (26) we would obtain terms of the form

$$\cos(\Omega t + \Theta_1) \quad , \quad \exp[-\zeta\omega_s t]\cos(\sqrt{1.082 - \zeta 2}\,\omega_s t + \Theta_2) \quad ,$$

and

$$\exp[-\zeta\omega_s t]\cos(\sqrt{0.9241 - \zeta 2}\,\omega_s t + \Theta_3) \quad ,$$

in which

$$\zeta = \frac{K}{2\omega_s} \quad .$$

From Eq. (27) we would obtain terms of the form

$$\cos(\Omega t + \Theta_1') \quad , \quad \exp[-\zeta\omega_s t]\cos(\sqrt{1 - \zeta 2}\,\omega_s t + \Theta_2')$$

and

$$\exp[-\zeta\omega_s t]\,\omega_s t\cos(\sqrt{1 - \zeta 2}\,\omega_s t + \Theta_3') \quad .$$

Now, $\cos A - \cos B = 2 \cos 1/2\,(A - B)$; hence, the terms from Eq. (26) could be alternatively expressed approximately as

$$\cos(\Omega t + \Theta_1) \quad , \quad \exp[-\zeta\omega_s t] \cos(\sqrt{1 - \zeta 2}\ \omega_s t + \Theta_2'')$$

and

$$\exp[-\zeta\omega_s t] \cos(\sqrt{1 - \zeta 2}\ \omega_s t + \Theta_2''') \cdot \sin[(\sqrt{1.082 - \zeta 2} - \sqrt{0.9241 - \zeta 2})\ \omega_s t] \quad .$$

Now, $\sin t \approx t$ for small t; thus, the terms from Eqs. (27) and (26) will be nearly the same for $t < 84$ minutes. This is true regardless of the value of the damping ratio ζ. Now, if $\zeta > 1/2$, then by the time that t no longer approximates $\sin \omega_{beat} t$, the quantity $\exp[-\zeta\omega_s t]$ is so small that these terms can be neglected anyway. Thus, for

$\zeta < 1/2$ and all times t

or

$t < 84$ minutes and all ζ ,

we have

$$D \approx (S^2 + \Omega^2)^2 (S^2 + 2\zeta\omega_s S + \omega_s^2)^2 \quad . \tag{28}$$

Under this assumption [Eq. (28)] the transfer functions of interest are (Fig. 3),

$$H_{\Delta\lambda\delta A_x}(S) = \frac{1}{R} \frac{1}{(S + \zeta\omega_s)^2 + \omega_d^2} \quad ,$$

$$H_{\Delta\lambda\delta V_x}(S) = \frac{2\zeta\omega_s}{R[(S + \zeta\omega_s)^2 + \omega_d^2]} \quad ,$$

$$H_{\Delta\lambda\epsilon_x} = \frac{-\omega_2}{S^2 + \Omega^2}\, t_{11} + \frac{S^2 + \omega_1^2}{C_1 S(S^2 + \Omega^2)}\, t_{21}$$

$$= \frac{-\omega_2\omega_s^2}{(S^2 + \Omega^2)(S^2 + 2\zeta\omega_s S + \omega_s^2)} - \frac{2\omega_s^2\omega_2(S^2 + \omega_1^2)}{(S^2 + \Omega^2)(S^2 + 2\zeta\omega_s S + \omega_s^2)^2} \quad ,$$

$$H_{\Delta\lambda\epsilon_y}(S) = \frac{S}{S^2 + \Omega^2}\, t_{11} + \frac{\omega_2}{C_1(S^2 + \Omega^2)}\, t_{21}$$

$$= \frac{\omega_1\omega_s^2}{(S^2 + \Omega^2)(S^2 + 2\zeta\omega_s S + \omega_s^2)} - \frac{2\omega_1\omega_2^2\omega_s^2}{(S^2 + \Omega^2)(S^2 + 2\zeta\omega_s S + \omega_s^2)^2} \quad ,$$

$$H_{\Delta\lambda\epsilon_z}(S) = \frac{\omega_1}{S^2 + \Omega^2}\, t_{11} + \frac{\omega_1\omega_2}{SC_1(S^2 + \Omega^2)}\, t_{21}$$

$$= \frac{\omega_1\omega_s^2}{(S^2 + \Omega^2)(S^2 + 2\zeta\omega_s S + \omega_s^2)} - \frac{2\omega_1\omega_2^2\omega_s^2}{(S^2 + \Omega^2)(S^2 + 2\zeta\omega_s S + \omega_s^2)^2} \quad ,$$

$$H_{\Delta\Lambda\delta A_y} = -\frac{1}{C_1} H_{\Delta\lambda\delta A_x} \qquad H_{\Delta\Lambda\delta V_x} = \frac{-1}{C_1} H_{\Delta\lambda\delta V_x} \quad ,$$

$$H_{\Delta\Lambda\epsilon_x} = \frac{\omega_1\omega_2 t_{22}}{C_1 S(S^2+\Omega^2)} + \frac{\omega_1 t_{12}}{S^2+\Omega^2}$$

$$= \frac{\Omega\omega_2\omega_s^2}{S(S^2+\Omega^2)(S^2+2\zeta\omega_s S+\omega_s^2)} + \frac{2\omega_s^2\Omega\omega_2 S}{(S^2+\Omega^2)(S^2+2\zeta\omega_s S+\omega_s^2)^2}$$

$$H_{\Delta\Lambda\epsilon_y} = \frac{S t_{12}}{S^2+\Omega^2} + \frac{\omega_2 t_{22}}{C_1(S^2+\Omega^2)}$$

$$= \frac{\omega_2\omega_s^2}{C_1(S^2+\Omega^2)(S^2+2\zeta\omega_s S+\omega_s^2)} + \frac{2\omega_s^2\omega_2 S^2}{(S^2+\Omega^2)(S^2+2\zeta\omega_s S+\omega_s^2)^2} \quad ,$$

$$H_{\Delta\Lambda\epsilon_z} = \frac{\omega_2 t_{12}}{S^2+\Omega^2} + \frac{(S^2+\omega_1^2)}{C_1 S(S^2+\Omega^2)}\, t_{22}$$

$$= \frac{\omega_s^2(S^2+\omega_1^2)}{C_1 S(S^2+\Omega^2)(S^2+2\zeta\omega_s S+\omega_s^2)} - \frac{2\omega_s^2\omega_2^2 S}{(S^2+\Omega^2)(S^2+2\zeta\omega_s S+\omega_s^2)^2} \quad ,$$

$$H_{\Delta\lambda\delta_y}(S) = \frac{2\omega_2}{R}\,\frac{S}{(S^2+2\zeta\omega_s S+\omega_s^2)^2} = C_1 H_{\Delta\Lambda\delta A_x}(S) \quad ,$$

$$H_{\Delta\lambda\delta V_y}(S) = \frac{2\omega_2 K}{R}\,\frac{S}{(S^2+2\zeta\omega_s S+\omega_s^2)^2} = C_1 H_{\Delta\Lambda\delta V_x}(S) \quad ,$$

in which

$$2\zeta\omega_s = K \quad , \quad \omega_d^2 = (1-\zeta^2)\,\omega_s^2 \quad .$$

We will also need to know the relationships among the variables ψ_x, ψ_y and ψ_z and the inputs ϵ_x, ϵ_y and ϵ_z. Transforming Eqs. (23), (24) and (25) and solving, we obtain:

$$\psi_x = \frac{S^2+\omega_1^2}{S(S^2+\Omega^2)}(\epsilon_x+\psi_{xo}) + \frac{\omega_2}{S^2+\Omega^2}(\epsilon_y+\psi_{yo}) + \frac{\omega_1\omega_2}{S(S^2+\Omega^2)}(\epsilon_z+\psi_{zo}) \quad , \tag{29}$$

$$\psi_y = \frac{-\omega_2}{(S^2+\Omega^2)}(\epsilon_x+\psi_{xo}) + \frac{S}{S^2+\Omega^2}(\epsilon_y+\psi_{yo}) + \frac{\omega_1}{S^2+\Omega^2}(\epsilon_z+\psi_{zo}) \quad , \tag{30}$$

$$\psi_z = \frac{\omega_1\omega_2}{S(S^2+\Omega^2)}(\epsilon_x+\psi_{xo}) - \frac{\omega_1}{S^2+\Omega^2}(\epsilon_y+\psi_{yo}) + \frac{S^2+\omega_2^2}{S(S^2+\Omega^2)}(\epsilon_z+\psi_{zo}) \quad , \tag{31}$$

in which ψ_{xo}, ψ_{yo} and ψ_{zo} denote the initial values of ψ_x, ψ_y and ψ_z, respectively.

E. Time-Domain Responses

Since our ultimate goal is to find a sampled-data model to represent the manner in which the error propagates between discrete observation times, we will need the impulse responses associated with the transfer functions obtained in Sec. III-D.

Let

$$F_1(S) = \frac{1}{S(S^2 + \Omega^2)\,(S^2 + 2\zeta\omega_s S + \omega_s^2)}$$

and

$$F_2(S) = \frac{1}{(S^2 + \Omega^2)\,(S^2 + 2\zeta\omega_s S + \omega_s^2)^2} \quad .$$

We can obtain the desired impulse responses by means of the six transform pairs listed below:

$$F_1(S) \leftrightarrow \frac{b}{\Omega^2} + \frac{d}{\omega_s^2} + \frac{2}{\Omega}\sin\Omega t - \frac{b}{\Omega^2}\cos\Omega t$$

$$+ \exp[-\zeta\omega_s t]\left(\frac{C\omega_s - \zeta d}{\omega_s\omega_d}\sin\omega_d t - \frac{d}{\omega_s^2}\cos\omega_d t\right) \quad t \geqslant 0 \quad ,$$

$$SF_1(S) \leftrightarrow a\cos\Omega t + \frac{b}{\Omega}\sin\Omega t$$

$$+ \exp[-\zeta\omega_s t]\left(c\cos\omega_d t + \frac{d - c\zeta\omega_s}{\omega_d}\sin\omega_d t\right) \quad t \geqslant 0 \quad ,$$

$$S^2F_1(S) \leftrightarrow -a\Omega\sin\Omega t + b\cos\Omega t$$

$$+ \exp[-\zeta\omega_s t]\left\{(d - 2c\zeta\omega_s)\cos\omega_d t - \frac{1}{\omega_d}[c\omega_s^2(1 - 2\zeta^2)\right.$$

$$\left. + c\zeta\omega_s]\sin\omega_d t\right\} \quad t \geqslant 0 \quad ,$$

in which

$$a = \frac{-2\zeta[1 + 2(1 - \zeta^2)\,(\Omega^2/\omega_s^2)]}{\omega_s^3} \quad ,$$

$$b = \frac{1 + (1 - 2\zeta^2)\,(\Omega^2/\omega_s^2)}{\omega_s^2} \quad ,$$

$$c = -a \quad ,$$

$$d = \frac{4\zeta^2 - 1 - (1 - 10\zeta^2 + 8\zeta^4)\,(\Omega^2/\omega_s^2)}{\omega_s^2} \quad ,$$

and

$$F_2(S) \leftrightarrow A \cos \Omega t + \frac{B}{\Omega} \sin \Omega t + \frac{\exp[-\zeta\omega_s t]}{2\omega_d^3} (\textcircled{1} \sin\omega_d t + 2 \cos\omega_d t$$

$$+ \textcircled{3}\, t \sin\omega_d t + \textcircled{4}\, t \cos\omega_d t) \qquad t \geqslant 0 \quad ,$$

$$SF_2(S) \leftrightarrow -A\Omega \sin \Omega t + B \cos \Omega t + \frac{\exp[-\zeta\omega_s t]}{2\omega_d^3} (\textcircled{5} \sin\omega_d t + \textcircled{6} \cos\omega_d t$$

$$+ \textcircled{7}\, t \sin\omega_d t + \textcircled{8}\, t \cos\omega_d t) \qquad t \geqslant 0 \quad ,$$

$$S^2F_2(S) \leftrightarrow -A\Omega^2 \cos \Omega t - B\Omega \sin \Omega t + \frac{\exp[-\zeta\omega_s t]}{2\omega_d^3} [(-\zeta\omega_s \textcircled{5} + \textcircled{7} - \omega_d \textcircled{6}) \sin\omega_d t$$

$$+ (-\zeta\omega_s \textcircled{6} + \textcircled{8} + \omega_d \textcircled{5}) \cos\omega_d t + (-\zeta\omega_s \textcircled{7} - \omega_d \textcircled{8})\, t \sin\omega_d t$$

$$+ (-\zeta\omega_s \textcircled{8} + \omega_d \textcircled{7})\, t \cos\omega_d t] \qquad t \geqslant 0 \quad ,$$

in which

$$\textcircled{1} = \frac{1}{\omega_s^2} \left[-3 + 12\zeta^2 - 8\zeta^4 + \frac{\Omega^2}{\omega_s^2} (-5 + 60\zeta^2 - 120\zeta^4 + 64\zeta^6) \right] \quad ,$$

$$\textcircled{2} = \frac{8\omega_d^3\zeta}{\omega_s^5} \left[1 + (3 - 8\zeta^2) \frac{\Omega^2}{\omega_s^2} \right] \quad ,$$

$$\textcircled{3} = \frac{2\zeta\omega_d^2}{\omega_s^3} \left[1 + \frac{\Omega^2}{\omega_s^2} (2 - 4\zeta^2) \right] \quad ,$$

$$\textcircled{4} = \frac{\omega_d}{\omega_s^2} \left[1 - 2\zeta^2 + \frac{\Omega^2}{\omega_s^2} (1 - 8\zeta^2 + 8\zeta^4) \right] \quad ,$$

$$\textcircled{5} = \frac{\zeta}{\omega_s} \left[-3 + 2\zeta^2 + \frac{\Omega^2}{\omega_s^2} (-15 + 40\zeta^2 - 24\zeta^4) \right] \quad ,$$

$$\textcircled{6} = -\frac{2\omega_d^3}{\omega_s^4} \left[1 + \frac{\Omega^2}{\omega_s^2} (2 - 12\zeta^2) \right] \quad ,$$

$$\textcircled{7} = \frac{\omega_d^2}{\omega_s^2} \left[-1 + \frac{\Omega^2}{\omega_s^2} (-1 + 4\zeta^2) \right] \quad ,$$

$$\textcircled{8} = \frac{\zeta\omega_d}{\omega_s} \left[1 + \frac{\Omega^2}{\omega_s^2} (3 - 4\zeta^2) \right] \quad .$$

Using these transforms, we obtain the required impulse responses. Where possible, terms of the order of Ω^2/ω_s^2 have been ignored with respect to one. For certain values of the damping ratio ζ, some of the remaining terms in Ω^2/ω_s^2 may also be ignored. The reader should remember that, because of the approximations made in Sec. III-D, these expressions, which follow, are good for $t < 84$ minutes and all damping ratios, or all time and damping ratios greater than 1/2:

$$h_{\Delta\lambda\epsilon_x}(t) \approx \frac{2\zeta\omega_2}{\omega_s}\cos\Omega t - C_2\sin\Omega t + \exp[-\zeta\omega_s t]\left(-\sin\omega_d t\,\frac{\omega_2\omega_s^2}{\omega_d^3}\left\{\zeta^2(3-2\zeta^2)\right.\right.$$

$$\left.+\frac{\Omega^2}{\omega_s^2}[(-1+7\zeta^2-10\zeta^4+4\zeta^6)+C_2^2(3-12\zeta^2+8\zeta^4)]\right\} - \frac{2\omega_2\zeta}{\omega_s}\cos\omega_d t$$

$$\left.+\frac{\omega_2\omega_s^2}{\omega_d^2}t\cos\omega_d t + \frac{2\zeta\omega_2}{\omega_s\omega_d}\left[\omega_2^2 - \frac{\Omega^4}{\omega_s^2}(2-4\zeta^2)\right]t\sin\omega_d t\right) \qquad t \geqslant 0 \quad ,$$

$$h_{\Delta\lambda\epsilon_y}(t) \approx \frac{2\zeta\Omega}{\omega_s}\sin\Omega t + \cos\Omega t + \exp[-\zeta\omega_s t]\left(\frac{-\zeta\omega_s}{\omega_d}\sin\omega_d t - \cos\omega_d t\right.$$

$$\left.+\frac{\omega_2^2}{\omega_d}t\sin\omega_d t - \frac{\omega_2^2\zeta\omega_s}{\omega_d^2}t\cos\omega_d t\right) \qquad t \geqslant 0 \quad ,$$

$$h_{\Delta\lambda\epsilon_z}(t) \approx \frac{-2\zeta\omega_1}{\omega_s}\cos\Omega t + C_1\sin\Omega t + \exp[-\zeta\omega_s t]\left(\frac{\omega_1\omega_s^2}{\omega_d^3}\left\{(1-\zeta^2)(2\zeta^2-1)\right.\right.$$

$$\left.+\frac{\Omega^2}{\omega_s^2}[(-1+7\zeta^2-10\zeta^4+4\zeta^6) - C_2^2(-3+12\zeta^2-8\zeta^4)]\right\}\sin\omega_d t$$

$$+\frac{2\zeta\omega_1}{\omega_s}\cos\omega_d t - \frac{2\omega_2^2\omega_1\zeta}{\omega_d\omega_s}t\sin\omega_d t - \frac{\omega_2^2\omega_1}{\omega_d^2}[(1-2\zeta^2)$$

$$\left.+\frac{\Omega^2}{\omega_s^2}(1-8\zeta^2+8\zeta^4)]\,t\cos\omega_d t\right) \qquad t \geqslant 0 \quad ,$$

$$h_{\Delta\Lambda\epsilon_x}(t) \approx C_1\left(1+2\zeta^2\frac{\Omega^2}{\omega_s^2}\right) + \frac{C_2^2}{C_1}\cos\Omega t + \frac{2\zeta C_2^2\Omega}{C_1\omega_s}\sin\Omega t + \exp[-\zeta\omega_s t]\left(\frac{-\zeta\omega_s}{C_1\omega_d}\sin\omega_d t\right.$$

$$\left.-\frac{1}{C_1}\cos\omega_d t + \frac{\omega_2^2}{\omega_d C_1}t\sin\omega_d t - \frac{\zeta\omega_s\omega_2^2}{\omega_d^2 C_1}t\cos\omega_d t\right) \qquad t \geqslant 0 \quad ,$$

$$h_{\Delta\Lambda\epsilon_y}(t) \approx \frac{-2\zeta\omega_2}{C_1\omega_s}\cos\Omega t + \frac{C_2}{C_1}\sin\Omega t + \exp[-\zeta\omega_s t]\left\{\frac{\omega_2\omega_s^2}{C_1\omega_d^3}\left[\zeta^2(3-2\zeta^2)\right.\right.$$

$$\left. + \frac{\Omega^2}{\omega_s^2}(2-5\zeta^2-2\zeta^4+4\zeta^6)\right]\sin\omega_d t + \frac{2\zeta\omega_2}{C_1\omega_s}\cos\omega_d t$$

$$\left. - \frac{2C_2\Omega^3\zeta}{\omega_s\omega_d C_1}t\sin\omega_d t - \frac{\omega_2\omega_s^2}{C_1\omega_d^3}t\cos\omega_d t\right\} \qquad t \geq 0 \quad ,$$

$$h_{\Delta\Lambda\epsilon_z}(t) \approx C_2(1+2\zeta^2-\Omega^2/\omega_s^2) - \frac{2\omega_2\zeta}{\omega_s}\sin\Omega t - C_2\left[1-(1+2\zeta^2)\frac{\Omega^2}{\omega_s^2}\right]\cos\Omega t$$

$$+ \exp[-\zeta\omega_s t]\left(\frac{-\zeta^3(5-4\zeta^2)\,C_2\Omega^2\omega_s}{\omega_d^3}\sin\omega_d t - \frac{C_2\Omega^2(1+4\zeta^2)}{\omega_s^2}\cos\omega_d t\right.$$

$$\left. - \frac{C_2\Omega^2}{\omega_d}t\sin\omega_d t + \frac{C_2\zeta\Omega^2\omega_s}{\omega_d^2}t\cos\omega_d t\right) \qquad t \geq 0 \quad ,$$

$$h_{\Delta\lambda\delta A_x}(t) = \frac{1}{R\omega_s}\exp[-\zeta\omega_s t]\sin\omega_d t \qquad t \geq 0 \quad ,$$

$$h_{\Delta\lambda\delta V_x}(t) = \frac{2\zeta}{R}\exp[-\zeta\omega_s t]\sin\omega_d t \qquad t \geq 0 \quad ,$$

$$h_{\Delta\lambda\delta A_y}(t) = \frac{\omega_2}{R\omega_d^3}(-\zeta\omega_s\sin\omega_d t + \omega_d^2 t\sin\omega_d t + \zeta\omega_s\omega_d t\cos\omega_d t) \qquad t \geq 0 \quad ,$$

$$h_{\Delta\lambda\delta V_y}(t) = \frac{2\omega_2\zeta\omega_s}{R\omega_d^3}(-\zeta\omega_s\sin\omega_d t + \omega_d^2 t\sin\omega_d t + \zeta\omega_s\omega_d t\cos\omega_d t) \qquad t \geq 0 \quad ,$$

$$h_{\Delta\Lambda\delta A_y}(t) = -\frac{1}{C_1}h_{\Delta\lambda\delta A_x} \quad , \quad h_{\Delta\Lambda\delta V_y}(t) = -\frac{1}{C_1}h_{\Delta\lambda\delta V_x}(t) \quad ,$$

$$h_{\Delta\Lambda\delta A_x}(t) = \frac{1}{C_1}h_{\Delta\lambda\delta A_y}(t) \quad , \quad h_{\Delta\Lambda\delta V_x}(t) = \frac{1}{C_1}h_{\Delta\lambda\delta V_y}(t) \quad .$$

We will also need the response of the system to initial conditions. We use the notation

$$\frac{\psi_x(t)}{\psi_{x_o}}$$

to denote the response of $\psi_x(t)$ to an initial position of ψ_x of one unit. From Eqs. (29), (30) and (31) of Sec. III-D, we have

$$\frac{\psi_x(t)}{\psi_{x_o}} = h_{\psi_x\epsilon_x}(t) = C_1^2 + C_2^2\cos\Omega t \qquad t \geq 0 \quad ,$$

$$\frac{\psi_x(t)}{\psi_{y_o}} = h_{\psi_x\epsilon_y}(t) = C_2 \sin\Omega t \qquad t \geqslant 0 \quad ,$$

$$\frac{\psi_x(t)}{\psi_{z_o}} = h_{\psi_x\epsilon_z}(t) = C_1C_2(1 - \cos\Omega t) \qquad t \geqslant 0 \quad ,$$

$$\frac{\psi_y(t)}{\psi_{x_o}} = h_{\psi_y\epsilon_x}(t) = -C_2 \sin\Omega t \qquad t \geqslant 0 \quad ,$$

$$\frac{\psi_y(t)}{\psi_{y_o}} = h_{\psi_y\epsilon_y}(t) = \cos\Omega t \qquad t \geqslant 0 \quad ,$$

$$\frac{\psi_y(t)}{\psi_{z_o}} = h_{\psi_y\epsilon_z}(t) = C_1 \sin\Omega t \qquad t \geqslant 0 \quad ,$$

$$\frac{\psi_z(t)}{\psi_{x_o}} = h_{\psi_z\epsilon_x}(t) = C_1C_2(1 - \cos\Omega t) \qquad t \geqslant 0 \quad ,$$

$$\frac{\psi_z(t)}{\psi_{y_o}} = h_{\psi_z\epsilon_y}(t) = -C_1 \sin\Omega t \qquad t \geqslant 0 \quad ,$$

$$\frac{\psi_z(t)}{\psi_{z_o}} = h_{\psi_z\epsilon_z}(t) = C_2^2 + C_1^2 \cos\Omega t \qquad t \geqslant 0 \quad .$$

From the fact that

$$\frac{\psi_i}{\psi_{j_o}} = h_{\psi_i\epsilon_j}(t) \qquad i, j = x, y, z \quad ,$$

and the fact that ψ_x, ψ_y and ψ_z appear only as drives in the two differential equations for $\Delta\lambda$ and $\Delta\Lambda$ [Eqs. (15) and (16)], it follows that

$$\frac{\Delta\lambda(t)}{\psi_{i_o}} = h_{\Delta\lambda\epsilon_i}(t) \qquad i = x, y, z \quad ,$$

and

$$\frac{\Delta\Lambda(t)}{\psi_{i_o}} = h_{\Delta\Lambda\epsilon_i}(t) \qquad i = x, y, z \quad .$$

Our time-domain description of the system now lacks only the sixteen quantities relating $\Delta\lambda$, $\Delta\dot{\lambda}$, $\Delta\Lambda$ and $\Delta\dot{\Lambda}$ to $\Delta\Lambda_o$, $\Delta\dot{\Lambda}_o$, $\Delta\lambda_o$ and $\Delta\dot{\lambda}_o$. If we transform Eqs. (15) and (16), including the terms $\Delta\lambda_o$, etc., and solve, then by using the approximation for D used in Sec. III-D we obtain

$$\frac{\Delta\lambda(t)}{\Delta\lambda_o} = \frac{\Delta\Lambda(t)}{\Delta\Lambda_o} = \exp[-\zeta\omega_s t] \left(\cos\omega_d t + \frac{\zeta\omega_s}{\omega_d} \sin\omega_d t\right) \qquad t \geqslant 0 \quad ,$$

$$\frac{\Delta\lambda(t)}{\Delta\dot{\lambda}_o} = \frac{\Delta\Lambda(t)}{\Delta\dot{\Lambda}_o} = \frac{1}{\omega_d} \exp[-\zeta\omega_s t] \sin\omega_d t \qquad t \geq 0 \quad ,$$

$$\frac{\Delta\lambda(t)}{\Delta\Lambda_o} = -C_1^2 \frac{\Delta\Lambda(t)}{\Delta\lambda_o} = \frac{\omega_s^2 C_1 C_2 \Omega}{\omega_d^3} \exp[-\zeta\omega_s t] (\sin\omega_d t - \omega_d t \cos\omega_d t) \qquad t \geq 0 \quad ,$$

$$\frac{\Delta\lambda(t)}{\Delta\dot{\Lambda}_o} = -C_1^2 \frac{\Delta\Lambda(t)}{\Delta\dot{\lambda}_o} = \frac{-C_1 C_2 \Omega}{\omega_d^2} \exp[-\zeta\omega_s t] [-\zeta\omega_s \sin\omega_d t + t(\omega_d^2 \sin\omega_d t$$

$$+ \zeta\omega_s\omega_d \cos\omega_d t)] \qquad t \geq 0 \quad .$$

Also,

$$\left(\frac{\Delta\dot{\lambda}(t)}{\cdot}\right) = \frac{d}{dt}\left(\frac{\Delta\lambda(t)}{\cdot}\right) \quad , \quad \text{etc.}$$

F. Approximate Time-Domain Response for $t \leq 84$ Minutes

The time responses given in the preceding section (III-E) are cumbersome, to say the least. If we consider times up to 84 minutes and are willing to make suitable approximations, these time responses can be considerably simplified. There are two situations in which this short time approximation is applicable:

(1) When the length of time of the whole operation does not exceed 84 minutes.

(2) When by means of external measurements each of our seven variables λ, $\dot{\lambda}$, Λ, $\dot{\Lambda}$, ψ_x, ψ_y and ψ_z is determined independently to an accuracy comparable to that which the navigator can maintain for 84 minutes; a set of such measurements is then used at least every 84 minutes to reset the navigator.

Below are listed the approximate time responses; those not listed can be assumed to be negligible with respect to the given terms. These approximations have ignored terms of Ω/ω_s ($\Omega/\omega_s \approx 1/17$) or smaller with respect to one.

$$h_{\Delta\lambda\epsilon_y}(t) = \frac{\Delta\lambda(t)}{\psi_{y_o}} = 1 - \exp[-\zeta\omega_s t] (\cos\omega_d t + \frac{\zeta\omega_s}{\omega_d} \sin\omega_d t) \qquad t \geq 0 \quad ,$$

$$h_{\Delta\lambda\delta V_x}(t) = \frac{2\zeta}{R} \exp[-\zeta\omega_s t] \sin\omega_d t \qquad t \geq 0 \quad ,$$

$$h_{\Delta\lambda\delta A_x}(t) = \frac{1}{R\omega_s} \exp[-\zeta\omega_s t] \sin\omega_d t \qquad t \geq 0 \quad ,$$

$$h_{\Delta\Lambda\epsilon_x}(t) = \frac{\Delta\Lambda(t)}{\psi_{x_o}} = \frac{1}{C_1} h_{\Delta\lambda\epsilon_y}(t) \qquad t \geq 0 \quad ,$$

$$h_{\Delta\Lambda\delta V_y}(t) = -\frac{2\zeta}{C_1 R} \exp[-\zeta\omega_s t] \sin\omega_d t \qquad t \geq 0 \quad ,$$

$$h_{\Delta\Lambda\delta A_y}(t) = -\frac{1}{RC_1\omega_s} \exp[-\zeta\omega_s t] \sin\omega_d t \qquad t \geq 0 \quad ,$$

$$h_{\psi_x\epsilon_x}(t) = \frac{\psi_x(t)}{\psi_{x_o}} = 1 \qquad t \geq 0 \quad ,$$

$$h_{\psi_y \epsilon_y}(t) = \frac{\psi_y(t)}{\psi_{y_o}} = 1 \quad t \geqslant 0 \quad ,$$

$$h_{\psi_z \epsilon_z}(t) = \frac{\psi_z(t)}{\psi_{z_o}} = 1 \quad t \geqslant 0 \quad ,$$

$$\frac{\Delta\lambda(t)}{\Delta\lambda_o} = \exp[-\zeta\omega_s t]\,(\cos\omega_d t + \frac{\zeta\omega_s}{\omega_d}\sin\omega_d t) \quad t \geqslant 0 \quad ,$$

$$\frac{\Delta\lambda(t)}{\Delta\dot{\lambda}_o} = \frac{1}{\omega_d}\exp[-\zeta\omega_s t]\,\sin\omega_d t \quad t \geqslant 0 \quad ,$$

$$\frac{\Delta\Lambda(t)}{\Delta\Lambda_o} = \frac{\Delta\lambda(t)}{\Delta\lambda_o} \qquad \frac{\Delta\Lambda(t)}{\Delta\dot{\Lambda}_o} = \frac{\Delta\lambda(t)}{\Delta\dot{\lambda}_o} \quad ,$$

$$\frac{\Delta\dot{\lambda}(t)}{\Delta\lambda_o} = \frac{\Delta\dot{\Lambda}(t)}{\Delta\Lambda_o} = \frac{d}{dt}\left[\frac{\Delta\lambda(t)}{\Delta\lambda_o}\right] \qquad \frac{\Delta\dot{\lambda}(t)}{\Delta\dot{\lambda}_o} = \frac{\Delta\dot{\Lambda}(t)}{\Delta\dot{\Lambda}_o} = \frac{d}{dt}\left[\frac{\Delta\lambda(t)}{\Delta\dot{\lambda}_o}\right] \quad .$$

G. Equivalent Discrete-Time System

Using the results of Sec. III-E or III-F (whichever set of responses is applicable), we can now write the set of matrix equations which describes the error propagation between sample (observation) times. Let x(t) denote the seven-tuple column vector:

$$\Delta\lambda(t) = x_1$$

$$\Delta\dot{\lambda}(t) = x_2$$

$$\Delta\Lambda(t) = x_3$$

$$\Delta\dot{\Lambda}(t) = x_4$$

$$\psi_x(t) = x_5$$

$$\psi_y(t) = x_6$$

$$\psi_z(t) = x_7 \quad .$$

Then,

$$\underline{x}(t + T) = \Phi_{77}x(t) + \sum_{i=1}^{7} x^i(t + T) \quad , \tag{32}$$

in which

$$
\Phi_{77} = \begin{bmatrix}
\frac{\Delta\lambda(T)}{\Delta\lambda_o} & \frac{\Delta\lambda(T)}{\Delta\dot{\lambda}_o} & \frac{\Delta\lambda(T)}{\Delta\Lambda_o} & \frac{\Delta\lambda(T)}{\Delta\dot{\Lambda}_o} & h_{\Delta\lambda\epsilon_x}(T) & h_{\Delta\lambda\epsilon_y}(T) & h_{\Delta\lambda\epsilon_z}(T) \\
\left.\frac{d}{dt}\frac{\Delta\lambda(t)}{\Delta\lambda_o}\right|_{t=T} & \text{etc.} & & & & & \\
\frac{\Delta\Lambda(T)}{\Delta\lambda_o} & \frac{\Delta\Lambda(T)}{\Delta\dot{\lambda}_o} & \frac{\Delta\Lambda(T)}{\Delta\Lambda_o} & \frac{\Delta\Lambda(T)}{\Delta\dot{\Lambda}_o} & h_{\Delta\Lambda\epsilon_x}(T) & h_{\Delta\Lambda\epsilon_y}(T) & h_{\Delta\Lambda\epsilon_z}(T) \\
\left.\frac{d}{dt}\frac{\Delta\Lambda(t)}{\Delta\lambda_o}\right|_{t=T} & \text{etc.} & & & & & \\
0 & 0 & 0 & 0 & h_{\psi_x\epsilon_x}(T) & h_{\psi_x\epsilon_y}(T) & h_{\psi_x\epsilon_z}(T) \\
0 & 0 & 0 & 0 & h_{\psi_y\epsilon_x}(T) & h_{\psi_y\epsilon_y}(T) & h_{\psi_y\epsilon_z}(T) \\
0 & 0 & 0 & 0 & h_{\psi_z\epsilon_x}(T) & h_{\psi_z\epsilon_y}(T) & h_{\psi_z\epsilon_z}(T)
\end{bmatrix} ,
$$

and the term which appears in the k^{th} row of $\underline{x}^j(t + T)$ is

$$x_k^j = \int_0^T h_{x_k y_j}(\tau)\, y_j(t + T - \tau)\, d\tau \quad ,$$

$$j, k = 1, 2, \ldots, 7 \quad ,$$

$$y_1 = \delta V_x,\ y_2 = \delta A_x,\ y_3 = \delta V_y,\ y_4 = \delta A_y,\ y_5 = \epsilon_x,\ y_6 = \epsilon_y,\ y_7 = \epsilon_z \quad .$$

It should be noted that the second and fourth rows of Φ_{77} are the derivatives of the first and third rows, respectively, the derivatives being evaluated at $t = T$ as indicated. Similarly, we note that

$$h_{\dot{x}_k y_j}(t) = \frac{d}{dt} h_{x_k y_j}(t) \qquad k = 1, 3 \qquad j = 1, 2, \ldots, 7 \quad .$$

Any impulses appearing in these derivatives must be retained in the impulse responses $h_{\Delta\dot{\lambda}y_j}(t)$ and $h_{\Delta\dot{\Lambda}y_j}(t)$.

Our ultimate goal is to be able to study the error propagation when we couple our inertial navigator to external measurements in some optimum fashion. This problem seems to admit of treatment most readily when viewed in terms of Kalman's formulation of the optimum filter problem. In order to make our model amenable to treatment via Kalman's method, we must be able to express the inputs $\underline{x}^i(t + T)$ as the responses of linear (discrete) systems to white noise. Since our estimation procedure will be based strictly on first-order statistics, we must ensure only that the matrices

$$R_{ij}(NT) = E\{\underline{x}_t^i (\underline{x}_{t+NT}^j)^T\} \qquad N = 0, \pm 1, \pm 2, \ldots$$

$$i, j = 1, 2, \ldots, 7$$

are the same for both our original inputs and the responses to the linear systems. We will assume that the inputs $\epsilon_x(t)$, $\epsilon_y(t)$, $\epsilon_z(t)$, $\delta V_x(t)$, $\delta A_x(t)$, $\delta V_y(t)$ and $\delta A_y(t)$ are all mutually statistically independent. Thus we need consider the matrices R_{ij} only for $i = j$, $i = 1, 2, \ldots, 7$.

Consider first x^5, x^6, x^7. We assume that

$$\phi_{\epsilon_i\epsilon_i}(\tau) = E\{\epsilon_i(t)\epsilon_i(t+\tau)\} = \sigma_{\epsilon_i}^2 \exp[-\nu_i|\tau|] \qquad i = x, y, z \quad . \tag{33}$$

Thus, the j-kth entry of $R_{\epsilon_i\epsilon_i}(NT)$ is

$$r_{jk}^i(NT) = \int_0^T \int_0^T h_{x_j\epsilon_i}(\varphi_1)\, h_{x_k\epsilon_i}(\varphi_2)\, \phi_{\epsilon_i\epsilon_i}(NT + \varphi_1 - \varphi_2)\, d\varphi_1 d\varphi_2$$

$$= \begin{cases} \sigma_{\epsilon_i}^2 \left| H_{x_j\epsilon_i} H_{x_k\epsilon_i} \right| & N = 0 \\ \sigma_{\epsilon_i}^2 H_{x_j\epsilon_i}^- H_{x_k\epsilon_i}^+ \exp[-\nu_i|NT|] & N > 0 \\ \sigma_{\epsilon_i}^2 H_{x_j\epsilon_i}^+ H_{x_k\epsilon_i}^- \exp[-\nu_i|NT|] & N < 0 \quad , \end{cases} \tag{34}$$

in which

$$\left| H_{x_j\epsilon_i} H_{x_k\epsilon_i} \right| = \int_0^T \int_0^T h_{x_j\epsilon_i}(\varphi_1)\, h_{x_k\epsilon_i}(\varphi_2) \exp[-\nu_i|\varphi_1 - \varphi_2|]\, d\varphi_1 d\varphi_2 \quad , \tag{35}$$

$$H_{x_j\epsilon_i}^+ = \int_0^T h_{x_j\epsilon_i}(\varphi_1) \exp[+\nu_i\varphi_1]\, d\varphi_1 \quad , \tag{36}$$

$$H_{x_j\epsilon_i}^- = \int_0^T h_{x_j\epsilon_i}(\varphi_1) \exp[-\nu_i\varphi_1]\, d\varphi_1 \quad , \tag{37}$$

$$i = x, y, z \qquad j, k = 1, 2, \ldots, 7 \quad ,$$

$$x_1 = \Delta\lambda,\ x_2 = \Delta\lambda,\ x_3 = \Delta\Lambda,\ x_4 = \Delta\Lambda,\ x_5 = \psi_x,\ x_6 = \psi_y,\ y_7 = \psi_z \quad .$$

Now, consider the representation

$$x_j^{\epsilon_i}(t) = [w_{j\epsilon_i}(t) + B_{ji}d_i(t - T)] \quad , \tag{38}$$

in which

$$d_i(t) = a_i d_i(t - T) + w_i(t) \quad ,$$

or

$$d_i(t) = \sum_{m=0}^{\infty} w_i(t - mT)\,(a_i)^m \qquad i = x, y, z \quad . \tag{39}$$

The w's are all white random processes with zero mean, and the w_i's are taken to have unit variance. The variance of the $w_{j\epsilon_i}$'s and the correlation between all the w's are left free for the time being.

Substituting Eq. (39) into Eq. (38) and noting that

$$\overline{w_i(t)w_j(t + NT)} = \overline{w_i w_j}\,\delta(NT) \quad , \tag{40}$$

we obtain

$$\overline{x_j^{\epsilon_i}(t)x_k^{\epsilon_i}(t + NT)} = r_{jk}^{i}(NT) \quad ,$$

$$= \begin{cases} \left(\overline{w_{j\epsilon_i} w_{k\epsilon_i}} + \dfrac{1}{1 - a_i^2} B_{ji}B_{ki}\right) & N = 0 \\ \left(B_{ki}a_i^{-1}\overline{w_{j\epsilon_i} w_i} + \dfrac{B_{ji}B_{ki}}{1 - a_i^2}\right) a^{|N|} & N > 0 \\ \left(B_{ji}a_i^{-1}\overline{w_{k\epsilon_i} w_i} + \dfrac{B_{ji}B_{ki}}{1 - a_i^2}\right) a^{|N|} & N < 0 \quad . \end{cases} \tag{41}$$

Our discrete white-noise-generated inputs will then match our original inputs if

$$a_i = e^{-\nu_i T} \quad , \tag{42}$$

$$\sigma_{\epsilon_i}^2 \left| H_{j\epsilon_i} H_{k\epsilon_i} \right| = \overline{w_{j\epsilon_i} w_{k\epsilon_i}} + \frac{1}{1 - a_i^2} B_{ji}B_{ki} \quad , \tag{43}$$

$$\sigma_{\epsilon_i}^2 H_{j\epsilon_i}^{-} H_{k\epsilon_i}^{+} = B_{ki}a_i^{-1}\overline{w_{j\epsilon_i} w_i} + \frac{B_{ji}B_{ki}}{1 - a_i^2} \quad , \tag{44}$$

$$\sigma_{\epsilon_i}^2 H_{j\epsilon_i}^{+} H_{k\epsilon_i}^{-} = B_{ji}a_i^{-1}\overline{w_{k\epsilon_i} w_i} + \frac{B_{ji}B_{ki}}{1 - a_i^2} \quad . \tag{45}$$

The solutions to Eqs. (43), (44) and (45) are

$$B_{ji} = \sigma_{\epsilon_i} H_{j\epsilon_i}^{+} \quad , \tag{46}$$

$$\overline{w_i w_{j\epsilon_i}} = \sigma_{\epsilon_i}^2 \left(H_{j\epsilon_i}^{-} - \frac{H_{j\epsilon_i}^{+}}{1 - a_i^2} \right) \quad , \tag{47}$$

$$\overline{w_{j\epsilon_i} w_{k\epsilon_i}} = \sigma_{\epsilon_i}^2 \left(\left| H_{j\epsilon_i} H_{k\epsilon_i} \right| - \frac{H_{j\epsilon_i}^{+} H_{k\epsilon_i}^{+}}{1 - a_i^2} \right) \quad , \tag{48}$$

$$i = x, y, z \qquad j, k = 1, 2, \ldots, 7 \quad ,$$

as can easily be verified by substitution. We can thus obtain equivalent discrete inputs for the signals $\epsilon_x(t)$, $\epsilon_y(t)$ and $\epsilon_z(t)$.

We now consider the inputs $\delta V_x(t)$, $\delta A_x(t)$, $\delta V_y(t)$ and $\delta A_y(t)$. Our analysis here will be much easier, as we can assume that

3-22-4248

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{10}(t) \end{bmatrix}
=
\left[\begin{array}{c|ccc}
 & \sigma_{\epsilon_x} H^+_{x_1\epsilon_x} & \sigma_{\epsilon_y} H^+_{x_1\epsilon_y} & \sigma_{\epsilon_z} H^+_{x_1\epsilon_z} \\
 & \sigma_{\epsilon_x} H^+_{x_2\epsilon_x} & \sigma_{\epsilon_y} H^+_{x_2\epsilon_y} & \sigma_{\epsilon_z} H^+_{x_2\epsilon_z} \\
\Phi_{77} & & & \\
 & & \cdots\cdots & \\
 & \sigma_{\epsilon_x} H^+_{x_7\epsilon_x} & \sigma_{\epsilon_y} H^+_{x_7\epsilon_y} & \sigma_{\epsilon_y} H^+_{x_7\epsilon_z} \\
\hline
 & e^{-\nu_x T} & 0 & 0 \\
0 & 0 & e^{-\nu_y T} & 0 \\
 & 0 & 0 & e^{-\nu_z T}
\end{array}\right]
\begin{bmatrix} x_1(t-T) \\ x_2(t-T) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{10}(t-T) \end{bmatrix}
+
\begin{bmatrix} w_1(t-T) \\ w_2(t-T) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_{10}(t-T) \end{bmatrix}
$$

$$w_i(t-T) = \sum_{j=1}^{7} w_{ij}(t-T)$$

$$i = 1, 2, \ldots, 7$$

$$w_8(t-T) = w_x(t-T)$$

$$w_9(t-T) = w_y(t-T)$$

$$w_{10}(t-T) = w_z(t-T)$$

$$x_1 = \Delta\lambda,\ x_2 = \Delta\dot{\lambda},\ x_3 = \Delta A,\ x_4 = \Delta\dot{A}$$

$$x_5 = \phi_x,\ x_6 = \phi_y,\ x_7 = \phi_z$$

$$y_1 = \delta V_x,\ y_2 = \delta A_x,\ y_3 = \delta V_y,\ y_4 = \delta A_y$$

$$y_5 = \epsilon_x,\ y_6 = \epsilon_y,\ y_7 = \epsilon_z$$

$$\overline{w_{i,t}w_{j,t}} = \sum_{k=1}^{7} \overline{w_{ik,t}w_{jk,t}} \qquad i, j, = 1, 2, \ldots, 7$$

$$\overline{w_{i,t}w_{j,t}} = \sum_{k=1}^{7} \overline{w_{i,t}w_{jk,t}} \qquad i = 8, 9, 10;\quad j = 1, 2, \ldots, 7$$

$$\overline{w_{i,t}w_{j,t}} = \delta_{ij} \qquad i, j = 8, 9, 10$$

in which

$$\overline{w_{jm}w_{km}} = \sigma_m^2 \left|H_{jm}H_{km}\right| \qquad j, k = 1, 2, \ldots, 7;\quad m = 1, 2, 3, 4$$

$$\overline{w_{jm}w_{km}} = \sigma_m^2 \left[\left|H_{jm}H_{km}\right| - \frac{H^+_{jm}H^+_{km}}{1 - e^{-2\nu_m T}}\right] \qquad j, k, = 1, 2, \ldots 7;\quad m = 5, 6, 7$$

$$\overline{w_{j\epsilon_i}w_i} = \sigma_{\epsilon_i} e^{-\nu_i T}\left[H^-_{j\epsilon_i} - \frac{H^+_{j\epsilon_i}}{1 - e^{-2\nu_i T}}\right] \qquad j = 1, 2, \ldots, 7;\quad i = x, y, z$$

in which

$$H^+_{jm} = \int_0^T h_{x_j y_m}(\sigma)\, e^{+\nu_m \sigma}\, d\sigma$$

$$H^-_{jm} = \int_0^T h_{x_j y_m}(\sigma)\, e^{-\nu_m \sigma}\, d\sigma$$

$$\left|H_{jm}H_{km}\right| = \int_0^T \int_0^T h_{x_j y_m}(\sigma_1)\, h_{x_k y_m}(\sigma_2) \times e^{-\nu_m |\sigma_1 - \sigma_2|}\, d\sigma_1 d\sigma_2$$

Fig. 4. Summary of discrete-time error propagation. The components of Φ_{77} and the terms H^+_{jk}, H^-_{jk}, $|H_{jm}H_{km}|$ depend on position.

$$\phi_{y_i y_i}(\tau) = \varphi_i^2 \exp[-\nu_i |\tau|] \qquad i = 1, 2, 3, 4 \quad ,$$

$$y_1 = \delta V_x,\ y_2 = \delta A_x,\ y_3 = \delta V_y,\ y_4 = \delta A_y \quad ,$$

where ν_i is large enough so that

$$\int_o^T h_{x_k y_j}(\tau)\, y_j(t-\tau)\, d\tau$$

is uncorrelated with

$$\int_o^T h_{x_k y_j}(\tau)\, y_j(t + NT - \tau)\, d\tau$$

for

$$N \neq 0 \quad , \quad j = 1, 2, 3, 4, \ k = 1, 2, \ldots, 7 \quad .$$

Thus, these four inputs can be replaced by the discrete white inputs

$$w_{kj} \quad , \quad k = 1, 2, \ldots, 7;\ j = 1, 2, 3, 4 \quad ,$$

$$\overline{w_{kj} w_{nm}} = \sigma_k^2 \int_o^T \int_o^T h_{x_k y_j}(\varphi_1)\, h_{x_k y_j}(\varphi_2) \exp[-\nu_k |\varphi_1 - \varphi_2|]\, d\varphi_1 d\varphi_2 \quad ,$$

where

$$m = j \quad n = k$$

$$= 0 \quad \text{otherwise}$$

where

$$k = 1, 2, \ldots, 7$$

$$j = 1, 2, 3, 4 \quad .$$

Recall that we have assumed that δV_x, δV_y, δA_x and δA_y were statistically independent. In practice δV_x and δV_y will not be uncorrelated. This can easily be corrected by setting

$$\overline{w_{k\delta v_x} w_{j\delta v_y}} = \int_o^T \int_o^T h_{x_k \delta v_x}(\varphi_1)\, h_{x_j \delta v_y}(\varphi_2)\, R_{\delta v_x \delta v_y}(\varphi_1 - \varphi_2)\, d\varphi_1 d\varphi_2$$

$$j, k = 1, 2, \ldots, 7 \quad .$$

We now have all the information needed for our discrete-time model. This information is summarized in Fig. 4. For operating times less than 84 minutes the impulse responses of Sec. III-F may be used; otherwise, those of Sec. III-E must be used. For picket ship operation the components of Φ_{77} and the terms H_{jm}^+, H_{jm}^- and $|H_{jm} H_{km}|$ are constant. For normal marine operation the coefficients of the differential equations governing error propagation (15, 16, 23, 24 and 25) vary so little during a four-hour period that they can often be considered constant. Hence, for $T < 4$ hours, Φ_{77} and the quantities H_{jm}^+, H_{jm}^- and $|H_{jm} H_{km}|$ can often be determined at each sample time by using the given expressions with the appropriate values of C_1 and C_2 inserted into the impulse responses.

IV. ESTIMATION OF NAVIGATOR ERROR

A. Estimation Equations When Navigator Is Not Reset Following Each Observation

In developing our solution to the estimation problem we will make repeated use of two properties:

(1) Let x_1 and x_2 be two random variables, and let $\hat{x}_1$ and $\hat{x}_2$ be the best linear estimates of x_1 and x_2, respectively, using the <u>same</u> observed data for each estimate. Then, the best linear estimate of $ax_1 + bx_2$, again using the same data, is

$$\widehat{ax_1 + bx_2} = a\hat{x}_1 + b\hat{x}_2 \quad .$$

(2) Consider any fixed class F of functionals on some given data with the property that any linear combination of functionals in F is again in F. Let $\hat{x}$ be the functional in this class that is the best estimator of x. Then, $(x - \hat{x})$ is uncorrelated with any functional of the given class, that is,

$$E\{(x - \hat{x})\, f\} = 0 \qquad f \epsilon F,\ \hat{x} \epsilon F \quad .$$

In the above statements "best" is used in the minimum-mean-square error sense. Property (2) follows directly if we regard mean-square estimation as finding a projection in an inner product space. However, for completeness, simple proofs of both properties are given in the appendix.

For convenience we take T, the time between observations, to be one unit. Then the model for our process is

$$\underline{z}(t) = H(t)\underline{x}(t) + \underline{v}(t) \quad , \tag{49}$$

$$\underline{x}(t) = \Phi(t)\underline{x}(t-1) + \underline{w}(t-1) \quad , \tag{50}$$

in which $\underline{v}(t)$ and $\underline{w}(t)$ are white and have zero mean. Since it suits our present purpose, we also assume that $\underline{w}$ and $\underline{v}$ are uncorrelated. Now our estimator $\hat{\underline{x}}_t$ is linear and may be written

$$\hat{\underline{x}}(t) = \sum_{k=0}^{\infty} A(k)\underline{z}(t-k) \quad . \tag{51}$$

The sum in Eq. (51) actually extends back only over all past values of data; we use ∞ as the upper index for convenience. We now define

$$\underline{z}^*(t) = \underline{z}(t) - \sum_{k=1}^{\infty} \underline{B}_k \underline{z}(t-k)$$

such that $E\{\underline{z}^*_t z^T_{t-k}\} = 0$; $k = 1, 2, \ldots$. We can then express $\hat{\underline{x}}(t)$ as

$$\hat{\underline{x}}(t) = C(t)\, \underline{z}^*(t) + \sum_{k=1}^{\infty} C_k \underline{z}(t-k) \quad . \tag{52}$$

Now, because $z^*(t)$ is uncorrelated with all the $z(t-k)$, $k = 1, 2, \ldots$, the minimization equations separate into two sets of equations, one involving only C and the other involving only the C_k's. Thus, the C_k's can be determined independently of C and

$$\sum_{k=1}^{\infty} C_k \underline{z}(t-k) = \text{best linear estimate of } \underline{x}_t,$$
$$\text{given } \underline{z}(t-k), \quad k = 1, 2, \ldots \quad ;$$
$$= \text{best linear estimate of } \Phi(t)\underline{x}_{t-1} + \underline{w}_{t-1},$$
$$\text{given } \underline{z}(t-k), \quad k = 1, 2, \ldots \quad ;$$
$$= \Phi(t)\hat{\underline{x}}(t-1) + 0 \quad . \tag{53}$$

Equation (53) follows directly from property (1). Thus,

$$\hat{\underline{x}}(t) = C(t)\underline{z}^*(t) + \Phi(t)\hat{\underline{x}}(t-1) \quad . \tag{54}$$

Now, $\underline{z}^*(t)$ is uncorrelated with $\Phi(t)\hat{\underline{x}}(t-1)$;

$$\underline{z}^*(t) = \underline{z}(t) - B(t)\Phi(t)\hat{\underline{x}}(t-1) \quad ;$$

and

$$0 = \overline{\underline{z}^*{}_t[\Phi(t)\hat{\underline{x}}_{t-1}]^T}$$
$$= \overline{\underline{z}_t\hat{\underline{x}}_{t-1}^T}\,\Phi^T(t) - B(t)\Phi(t)\,\overline{\hat{\underline{x}}_{t-1}\hat{\underline{x}}_{t-1}^T}\,\Phi(t)^T \quad .$$

We assume that $\Phi(t)$ and $\overline{\hat{\underline{x}}_{t-1}\hat{\underline{x}}_{t-1}^T}$ are invertible; thus,

$$B(t) = \overline{\underline{z}_t\hat{\underline{x}}_{t-1}^T}\;\overline{\hat{\underline{x}}_{t-1}\hat{\underline{x}}_{t-1}^T}^{\,-1}\;\Phi^{-1}(t) \quad . \tag{55}$$

Now, from Eqs. (49) and (50)

$$\overline{\underline{z}_t\hat{\underline{x}}_{t-1}^T} = H(t)\,\overline{\underline{x}_t\hat{\underline{x}}_{t-1}^T} + \overline{\underline{v}_t\hat{\underline{x}}_{t-1}^T} = H(t)\,\overline{\underline{x}_t\hat{\underline{x}}_{t-1}^T}$$
$$= H(t)\left[\Phi(t)\,\overline{\underline{x}_{t-1}\hat{\underline{x}}_{t-1}^T} + \overline{\underline{w}_{t-1}\hat{\underline{x}}_{t-1}^T}\right]$$
$$= H(t)\,\Phi(t)\,\overline{\underline{x}_{t-1}\hat{\underline{x}}_{t-1}^T} \quad , \tag{56}$$

and from property (2) we have

$$\overline{\hat{\underline{x}}_{t-1}(\hat{\underline{x}}_{t-1}^T - \hat{\underline{x}}_{t-1}^T)} = 0 \quad ,$$

or

$$\overline{\hat{\underline{x}}_{t-1}\underline{x}_{t-1}^T} = \overline{\hat{\underline{x}}_{t-1}\hat{\underline{x}}_{t-1}^T} \quad . \tag{57}$$

Substituting Eqs. (57) and (56) into Eq. (55), we obtain

$$B(t) = H(t)\,\Phi(t)\,\overline{\hat{\underline{x}}_{t-1}\hat{\underline{x}}_{t-1}^T}\;\overline{\hat{\underline{x}}_{t-1}\hat{\underline{x}}_{t-1}^T}^{\,-1}\;\Phi(t)^{-1} = H(t) \tag{58}$$

and

$$\hat{\underline{x}}(t) = C(t)\,[\underline{z}(t) - H(t)\,\Phi(t)\,\hat{\underline{x}}(t-1)] + \Phi(t)\,\hat{\underline{x}}(t-1) \quad . \tag{59}$$

We now need to solve for C(t). From property (2) we have

$$\overline{(\underline{x}_t - \hat{\underline{x}}_t)\,\underline{z}_t^T} = 0 \quad ,$$

or

$$\overline{\{\underline{x}_t - C(t)[\underline{z}_t - H(t)\,\Phi(t)\,\hat{\underline{x}}_{t-1}] - \Phi(t)\,\hat{\underline{x}}_{t-1}\}\,\underline{z}_t^T} = 0 \quad ,$$

or

$$\overline{\underline{x}_t\underline{z}_t^T} - \Phi(t)\,\overline{\hat{\underline{x}}_{t-1}\underline{z}_t^T} = C(t)\left[\overline{\underline{z}_t\underline{z}_t^T} - H(t)\,\Phi(t)\,\overline{\hat{\underline{x}}_{t-1}\underline{z}_t^T}\right] \quad . \tag{60}$$

Now, by Eqs.(49) and (50), we have

$$\begin{aligned}
\overline{\hat{\underline{x}}_{t-1}\underline{z}_t^T} &= \overline{\hat{\underline{x}}_{t-1}\underline{x}_t^T}\;H^T(t) + \overline{\hat{\underline{x}}_{t-1}\underline{v}_t^T} \\
&= \overline{\hat{\underline{x}}_{t-1}\underline{x}_t^T}\;H^T(t) \\
&= \overline{\hat{\underline{x}}_{t-1}\underline{x}_{t-1}^T}\;\Phi^T(t)\;H^T(t) + \overline{\hat{\underline{x}}_{t-1}\underline{w}_{t-1}^T}\;H^T(t) \\
&= \overline{\hat{\underline{x}}_{t-1}\underline{x}_{t-1}^T}\;\Phi^T(t)\;H^T(t) \quad .
\end{aligned} \tag{61}$$

We can express the covariance matrix of the error as

$$\begin{aligned}
\Sigma(t-1) &= \overline{(\underline{x}_{t-1} - \hat{\underline{x}}_{t-1})\,(\underline{x}_{t-1}^T - \hat{\underline{x}}_{t-1}^T)} \\
&= \overline{(\underline{x}_{t-1} - \hat{\underline{x}}_{t-1})\,\underline{x}_{t-1}^T} - \overline{(\underline{x}_{t-1} - \hat{\underline{x}}_{t-1})\,\hat{\underline{x}}_{t-1}^T} \\
&= \overline{\underline{x}_{t-1}\underline{x}_{t-1}^T} - \overline{\hat{\underline{x}}_{t-1}\underline{x}_{t-1}^T} \quad ,
\end{aligned}$$

and thus

$$\overline{\hat{\underline{x}}_{t-1}\underline{x}_{t-1}^T} = \overline{\underline{x}_{t-1}\underline{x}_{t-1}^T} - \Sigma(t-1) \triangleq X(t-1) - \Sigma(t-1) \tag{62}$$

and

$$\overline{\hat{\underline{x}}_{t-1}\underline{z}_t^T} = X(t-1)\;\Phi^T(t)\;H^T(t) - \Sigma(t-1)\;\Phi^T(t)\;H^T(t) \quad . \tag{63}$$

From Eq.(49) we also obtain

$$\begin{aligned}
\overline{\underline{x}_t\underline{z}_t^T} &= \overline{\underline{x}_t\underline{x}_t^T}\;H^T(t) + \overline{\underline{x}_t\underline{v}_t^T} \\
&= X(t)\;H^T(t) \quad .
\end{aligned} \tag{64}$$

Also,

$$\overline{\underline{z}_t\underline{z}_t^T} = H(t)\ \overline{\underline{x}_t\underline{x}_t^T}H^T(t) + \overline{\underline{v}_t\underline{x}_t^T}H^T(t) + H(t)\ \overline{\underline{x}_t\underline{v}_t^T} + \overline{\underline{v}_t\underline{v}_t^T}$$

$$= H(t)\ X(t)\ H^T(t) + V(t) \quad , \tag{65}$$

in which V(t) denotes $\overline{\underline{v}_t\underline{v}_t^T}$. Further,

$$X(t) = \overline{\underline{x}_t\underline{x}_t^T} = \Phi(t)\ \overline{\underline{x}_{t-1}\underline{x}_{t-1}^T}\Phi^T(t) + \Phi(t)\ \overline{\underline{x}_{t-1}\underline{w}_{t-1}^T}$$

$$+ \overline{\underline{w}_{t-1}\underline{x}_{t-1}^T}\Phi^T(t) + \overline{\underline{w}_{t-1}\underline{w}_{t-1}^T}$$

$$= \Phi(t)\ X(t-1)\ \Phi^T(t) + W(t-1) \quad , \tag{66}$$

in which W(t) denotes $\overline{\underline{w}_t\underline{w}_t^T}$. Substituting Eqs. (63) through (66) into Eq. (60), we obtain

$$\Phi(t)\ X(t-1)\ \Phi^T(t)\ H^T(t) + W(t-1)\ H^T(t) - \Phi(t)\ X(t-1)\ \Phi^T(t)\ H^T(t) + \Phi(t)$$

$$\times \Sigma(t-1)\ \Phi^T(t)\ H^T(t) = C(t)\ [H(t)\ \Phi(t)\ X(t-1)\ \Phi^T(t)\ H^T(t) + H(t)\ W(t-1)$$

$$\times H^T(t) + V(t) - H(t)\ \Phi(t)\ X(t-1)\ \Phi^T(t)\ H^T(t) + H(t)\ \Phi(t)\ \Sigma(t-1)\ \Phi^T(t)\ H^T(t)]$$

and

$$C(t) = [W(t-1) + \Phi(t)\ \Sigma(t-1)\ \Phi^T(t)]\ H^T(t)\ \{V(t) + H(t)\ [W(t-1)$$

$$+ \Phi(t)\ \Sigma(t-1)\ \Phi^T(t)]\ H^T(t)\}^{-1} \quad . \tag{67}$$

Note that C(t) depends on $\Sigma(t-1)$; thus, in order to complete our recursive scheme, we must be able to express $\Sigma(t)$ in terms of C(t) and $\Sigma(t-1)$:

$$\Sigma(t) = \overline{(\underline{x}_t - \hat{\underline{x}}_t)\ (\underline{x}_t^T - \hat{\underline{x}}_t^T)} = \overline{\underline{x}_t(\underline{x}_t^T - \hat{\underline{x}}_t^T)}$$

$$= X(t) - \overline{\underline{x}_t\underline{z}_t^T}\ C^T(t) + \overline{\underline{x}_t\hat{\underline{x}}_{t-1}^T}\ \Phi^T(t)\ H^T(t)\ C^T(t) - \overline{\underline{x}_t\hat{\underline{x}}_{t-1}^T}\ \Phi^T(t) \quad . \tag{68}$$

Using Eqs. (64), (56) and (57), we obtain

$$\Sigma(t) = X(t)\ [I - H^T(t)\ C^T(t)] - \Phi(t)\ \overline{\hat{\underline{x}}_{t-1}\underline{x}_{t-1}^T}\ \Phi^T(t)\ [I - H^T(t)\ C^T(t)] \quad , \tag{69}$$

and using (62) and (66), we obtain

$$\Sigma(t) = \{\Phi(t)\ X(t-1)\ \Phi(t)^T + W(t-1) - \Phi(t)\ [X(t-1)$$

$$- \Sigma(t-1)]\ \Phi^T(t)\}\ [I - H^T(t)\ C^T(t)] \quad ,$$

$$\Sigma(t) = [W(t-1) + \Phi(t)\ \Sigma(t-1)\ \Phi^T(t)]\ [I - H^T(t)\ C^T(t)] \quad . \tag{70}$$

B. Estimation Equations When Navigator Is Reset Following Each Observation

The equations in Sec. IV-A were developed under the assumption that the estimate of the navigator error was not used to reset the quantities in the navigator. Here we assume that, following an observation, $\hat{\underline{x}}(t)$ is used to reset the navigator (either by resetting the navigator computer or both the computer and stable platform). If we use $\underline{\hat{x}}(t-1)$ to reset the navigator, then the model for our error propagation becomes

$$\underline{z}'(t) = H(t)\,\underline{x}'(t) + \underline{v}(t) \quad , \tag{71}$$

$$\underline{x}'(t) = \Phi(t)\,[\,\underline{x}'(t-1) - \underline{\hat{x}}'(t-1)] + \underline{w}(t-1) \quad , \tag{72}$$

in which we have used primes to distinguish the quantities in reset operation from those in normal operation. The relations between these two sets of quantities are:

$$\underline{x}'(0) = \underline{x}(0) \qquad \underline{\hat{x}}'(0) = \underline{\hat{x}}(0) = 0$$

$$\underline{x}'(1) = \underline{x}(1) \qquad \underline{\hat{x}}'(1) = \underline{\hat{x}}(1)$$

$$\underline{x}'(2) = \Phi_2[\,\underline{x}(1) - \underline{\hat{x}}(1)] + \underline{w}(1) = \underline{x}(2) - \Phi(2)\,\underline{\hat{x}}(1) \quad .$$

Thus,

$$\underline{\hat{x}}'(2) = \underline{\hat{x}}(2) - \Phi(2)\,\underline{\hat{x}}(1)$$

and

$$\begin{aligned}\underline{x}'(3) &= \Phi(3)\,[\,\underline{x}'(2) - \underline{\hat{x}}'(2) + \underline{w}(2) \\ &= \Phi(3)\,[\,\underline{x}(2) - \underline{\hat{x}}(2)] + \underline{w}(2) = \underline{x}(3) - \Phi(3)\,\underline{\hat{x}}(2) \quad .\end{aligned}$$

Thus,

$$\underline{\hat{x}}'(3) = \underline{\hat{x}}(3) - \Phi(3)\,\underline{\hat{x}}(2) \quad .$$

In general,

$$\underline{x}'(n) = \underline{x}(n) - \Phi(n)\,\underline{\hat{x}}(n-1) \quad ,$$

$$\underline{\hat{x}}'(n) = \underline{\hat{x}}(n) - \Phi(n)\,\underline{\hat{x}}(n-1)$$

and

$$\underline{z}'(n) = H(n)\,\underline{x}'(n) + V(n) = \underline{z}(n) - H(n)\,\Phi(n)\,\underline{\hat{x}}(n-1) \quad .$$

Substituting into

$$\underline{\hat{x}}(n) = C(n)\,[\,\underline{z}(n) - H(n)\,\Phi(n)\,\underline{\hat{x}}(n-1)] + \Phi(n)\,\underline{\hat{x}}(n-1) \quad ,$$

we obtain

$$\begin{aligned}\underline{\hat{x}}'(n) + \Phi(n)\,\underline{\hat{x}}(n-1) = C(n)\,[\,\underline{z}'(n) &+ H(n)\,\Phi(n)\,\underline{\hat{x}}(n-1) - H(n)\,\Phi(n)\,\underline{\hat{x}}(n-1)] \\ &+ \Phi(n)\,\underline{\hat{x}}(n-1) \quad ,\end{aligned}$$

or

$$\underline{\hat{x}}'(n) = C(n)\,\underline{z}'(n) \quad . \tag{73}$$

The quantity C(n) is obtained exactly as before, since

$$\begin{aligned}\Sigma'(n) &= \overline{(\underline{x}'_n - \underline{\hat{x}}'_n)\,(\underline{x}'_n - \underline{\hat{x}}'_n)^T} \\ &= \overline{(\underline{x}_n - \underline{\hat{x}}_n)\,(\underline{x}_n - \underline{\hat{x}}_n)^T} = \Sigma(n) \quad .\end{aligned}$$

The estimation scheme thus remains basically unchanged. The error is unchanged, as is the basic computational complexity, in that we must still generate the matrices C(t) and Σ(t) in the same recursive fashion. The estimator Eq. (73) is slightly simplified over Eq. (58).

C. Summary and General Remarks

In summary, we have the two possible models

$$\underline{z}(t) = H(t)\,\underline{x}(t) + \underline{v}(t) \qquad \text{[Eq. (49)]}$$

$$\underline{x}(t) = \Phi(t)\,\underline{x}(t-1) + \underline{w}(t-1) \quad , \qquad \text{[Eq. (50)]}$$

or

$$\underline{z}'(t) = H(t)\,\underline{x}'(t) + \underline{v}(t) \qquad \text{[Eq. (71)]}$$

$$\underline{x}'(t) = \Phi(t)\,[\underline{x}'(t-1) - \hat{\underline{x}}'(t-1)] + \underline{w}(t-1) \quad . \qquad \text{[Eq. (72)]}$$

The two estimators are given by

$$\hat{\underline{x}}(t) = C(t)\,[\underline{z}(t) - H(t)\,\Phi(t)\,\hat{\underline{x}}(t-1)] + \Phi(t)\,\hat{\underline{x}}(t-1) \qquad \text{[Eq. (59)]}$$

and

$$\hat{\underline{x}}'(t) = C(t)\,\underline{z}(t) \quad . \qquad \text{[Eq. (73)]}$$

The matrix C(t) is determined by the pair of recursive relations

$$C(t) = [W(t-1) + \Phi(t)\,\Sigma(t-1)\,\Phi^T(t)]\,H^T(t)\,\{V(t) + H(t)\,[W(t-1) + \Phi(t)\,\Sigma(t-1)\,\Phi^T(t)]\,H^T(t)\}^{-1} \qquad \text{[Eq. (67)]}$$

$$\Sigma(t) = [W(t-1) + \Phi(t)\,\Sigma(t-1)\,\Phi^T(t)]\,[I - H^T(t)\,C^T(t)] \quad , \qquad \text{[Eq. (70)]}$$

in which

$\Sigma(t)$ = covariance matrix of the estimation error $\underline{x}_t - \hat{\underline{x}}_t$

$W(t-1)$ = covariance matrix of the input $w(t-1)$

$V(t)$ = covariance matrix of the measurement error $v(t)$.

Our computation starts with some value for $\Sigma(o)$. If the platform were initially perfectly oriented and the initial position known exactly, then $\Sigma(o)$ would be zero. This will not usually be the case, and $\Sigma(o)$ will have to be determined from the alignment procedure used. Knowing $\Sigma(o)$, we calculate $C(1)$ from $\Sigma(o)$ via Eq. (67). Then $\Sigma(1)$ can be calculated from $C(1)$ via Eq. (70). This process can be repeated continuously and we obtain $\Sigma(t)$ for $t = 1, 2, \ldots$; in the process we have determined $C(t)$ and hence $\hat{x}(t)$. If $\Phi(t)$ is constant, i.e., $\Phi(t) \equiv \Phi$, then the steady-state value of Σ can possibly be obtained by setting $\Sigma(t) = \Sigma(t-1)$ and eliminating $C(t)$ between Eqs. (67) and (70).

This discussion concerning the steady-state value of $\Sigma(t)$ naturally poses the question: Under what conditions is the steady-state value of $\Sigma(t)$ unique? Another point that requires an answer is: When is the optimum estimator asymptotically stable? The asymptotic stability of the estimator assumes importance because we do not wish small bias inputs, which have been neglected in our analysis, to be able to cause arbitrarily large errors in our steady-state estimate. To these questions Kalman[5] provides the following answer, which is definitive but perhaps somewhat cumbersome to apply.

Theorem (Kalman):

Let the system defined by Eqs. (1) and (2) be completely observable and completely controllable. Then

(1) The optimum estimator is uniformly asymptotically stable.

(2) All solutions corresponding to different choices of a covariance matrix $\Sigma(o)$ converge uniformly to the same solution.

Kalman refers to the system defined by Eqs. (1) and (2) as completely observable if for every vector $\underline{P}$ and every time t_0 there exists a $T(t_0)$ and an unbiased estimator Π of $P^T x(T)$, in which $\hat{\Pi}$ is a linear function of $z(t)$, $t_0 \leqslant t \leqslant T$. He calls this same system completely controllable if there exists a forcing function $\underline{w}(t)$ which can take the system from rest to any desired state in a finite time. For mathematically definitive conditions for complete observability and complete controllability and a thorough discussion of the whole question, the reader is referred to Chapters 15 and 16 of Ref. 5.

One further point deserves mention. The estimator described gives simultaneously the minimum-mean-square error estimate of each component of $\underline{x}$. Our concern may center on the position estimate in some particular direction, say in the quantity $x_1 \cos\Theta + x_2 \sin\Theta$, and we may wonder if it is possible to find another estimator which gives a better estimate of $\cos\Theta\, x_1 + \sin\Theta\, x_2$. However, the estimate obtained using our estimator is $\cos\Theta\, \hat{x}_1 + \cos\Theta\, \hat{x}_2$, which, according to property (1), is the minimum-mean-square estimate of $\cos\Theta\, x_1 + \sin\Theta\, x_2$ for the given data. This property of our estimator is equivalent to a statement that the ellipsoid of concentration of our estimation error lies wholly within the ellipsoid of concentration obtainable by any other linear estimator.[1]

V. COMPUTATIONAL ASPECTS

In order to use the formulation presented in Sec. IV to find the optimum estimator or to solve for the minimum possible estimation error, considerable machine computation is necessary in almost all cases of interest. Some of the computations can be carried out on an analog computer, but the use of a digital machine at some points is unavoidable. We review here the basic quantities that must be calculated.

In order to apply the formulas given in Sec. IV there are five basic matrices which must be calculated: $\Phi(t)$, $W(t)$, $H(t)$, $V(t)$ and $\Sigma(o)$. The first two of these are found from the description of the inertial navigator, the second two from the description of the measurement process and the last from the description of the initial alignment process.

In obtaining the matrices $\Phi(t)$ and $W(t)$ there are 98 quantities which must be found. Forty-nine of these are the response at time T of each of the seven basic error quantities ($\Delta\Lambda$, $\Delta\dot{\Lambda}$, $\Delta\lambda$, $\Delta\dot{\lambda}$, ψ_x, ψ_y, ψ_z) to an initial unit displacement of each one of these same seven quantities. The second 49 quantities required are the responses at time T of each of the above seven quantities to a "unit impulse" excitation of each of the seven inputs (δV_x, δA_x, δV_y, δA_y, ϵ_x, ϵ_y, ϵ_z). These 98 quantities are given in Sec. III-E for all T. An approximate set, which can be used when the time of the total operation does not exceed 84 minutes, is given in Sec. III-F. Fortunately, not all of these 98 quantities are distinct; as pointed out in Sec. III-E, many of the quantities in the set of 49 initial condition responses are identical to quantities in the set of 49 impulse responses. If, for some reason, it is not desirable to use the responses given in Sec. III-E, they can be calculated in two ways. The first way would be to use a digital computer to numerically integrate out the responses in accordance with Eqs. (15), (16), (23), (24) and (25). An easier way might be to use an analog computer to simulate these five equations and find the desired responses by making the necessary number of runs on the analog computer. In finding the impulse responses, any input of duration less than $1/20^{th}$ of the smallest time constant of the system would be suitable.

As shown in Fig. 4, $\Phi(t)$ consists of a 7×7 matrix Φ_{77} plus several smaller arrays. The expression for Φ_{77} is given in Sec. III-E in terms of the initial condition responses mentioned above. Three of the remaining terms in $\Phi(t)$ are the time constants of the correlation functions of the gyro drifts and must be estimated. The other quantities in $\Phi(t)$ are the H^{+}'s. These must be calculated by the integration in Fig. 4. This integration could be carried out by means of either digital or analog equipment; it can also be performed analytically by hand, with perhaps a reasonable amount of patience.

As shown in Fig. 4, the entries of the W(t) matrix are found by calculating the quantities H^{+}, H^{-} and $|H_{jm}H_{km}|$. The quantities H^{-}, like the quantities H^{+}, are found by a single integration involving an impulse response, as indicated in Fig. 4. The quantities $|H_{jm}H_{km}|$ involve a double integration involving two impulse responses simultaneously. This double integration can be carried out on a digital machine and, perhaps somewhat awkwardly, on analog equipment. It can also be carried out by hand, but the task is so unwieldy that it would require an extreme amount of patience and devotion to duty.

We can only indicate here how the matrix H(t) is obtained, since we have not specified the method of taking measurements. We assumed that the measurements could be expressed as

$$\underline{r}(t) = \underline{F}[\underline{q}(t) + x(t)] + \underline{v}(t) \quad , \tag{74}$$

in which $\underline{q}(t)$ represents the true value of the seven quantities used to describe the navigator operation. The function $\underline{F}$ must be found from the geometry of the situation. The matrix H(t) is then expressible as

$$H_{ij}(t) = \left. \frac{\partial F_i(\underline{s}(t))}{\partial s_j(t)} \right|_{\underline{s}(t) = \underline{q}(t)} \quad . \tag{75}$$

Equation (75) can be evaluated analytically or by a numerical differencing approximation. The matrix V(t) must be found by an enlightened examination of the errors involved in the measurements.

The matrix $\Sigma(o)$ can be estimated only by someone well versed in the procedures used to align an inertial platform. For an introduction into the many possible means of alignment, the reader is referred to Chapter 6 of Ref. 4.

With the five matrices $\Phi(t)$, W(t), H(t), V(t) and $\Sigma(o)$ available, the recursive formulas summarized in Sec. IV-C can be employed. However, it should be mentioned that these formulas can be written in various ways and the best choice depends on the problem. The following is a rearrangement of the basic formula that is not self-evident. Define

$$I(t) = \Sigma^{-1}(t) \quad ; \tag{76}$$

then, in place of Eqs. (67) and (70) we have

$$I(t) = [\Phi(t)\, I^{-1}(t-1)\, \Phi^{T}(t) + W(t-1)]^{-1} + H^{T}(t)\, V^{-1}(t)\, H(t) \quad . \tag{77}$$

For the actual estimator, Eq. (59) can be replaced by

$$\begin{aligned} \hat{\underline{x}}(t) = I^{-1}(t)\{[\Phi(t)\, I^{-1}(t-1)\, \Phi^{T}(t) + W(t-1)]^{-1}\, \Phi(t)\, \hat{\underline{x}}(t-1) \\ + H^{T}(t)\, V^{-1}(t)\, \underline{z}(t)\} \quad . \end{aligned} \tag{78}$$

The necessary matrix manipulations needed to obtain Eqs. (77) and (78) are outlined in Statement 3, Appendix A.†

The formulas of Sec. IV require, at each step, the inversion of a matrix whose rank equals the number of measurements made (the number of components of z). Equations (77) and (78) on the other hand, require the inversion of a matrix with rank equal to the number of states (the number of components of x). In either case, matrix inversion is required and there are many possible techniques available. Reference 7 is one of many sources.

In the case in which Φ, W, H and V are constant, the steady-state value of Σ is the solution of the equation

$$\Sigma = F - FH^T[V + HFH^T]^{-1} HF \quad , \tag{79}$$

in which

$$F = W + \Phi\Sigma\Phi^T \quad .$$

This matrix equation results in a system of simultaneous quadratic equations which may be solvable by machine methods.

There is one potentially important technique for simplifying each recursive computation considerably while possibly increasing the number of recursive calculations. Recall that the damping ratio ζ is zero if the velocity log is not incorporated into the continuous system. With $\zeta = 0$, the expressions for H^+, H^-, $|H_{jm}H_{km}|$ and the entries of Φ_{77} simplify by almost an order of magnitude. The information given by the velocity log can still be used in correcting position errors by sampling the velocity-log signals at a rate twice the "bandwidth" of these signals. If the sampling time T of our system is set equal to the time between velocity-log samples, then these signals can be incorporated in $\underline{z}(t)$ and used in an optimum manner to help obtain the estimate of $\underline{x}(t)$. This procedure may also increase the over-all accuracy of the system, since the conventional velocity-log damping does not necessarily employ the velocity data in an optimum manner.

† The formulas can also be derived directly by using a line of development different from that in Sec. IV.

APPENDIX A
DERIVATION OF CERTAIN PROPERTIES OF THE OPTIMUM ESTIMATE AND AN ALTERNATE FORM FOR THE ESTIMATOR

Statement 1:— Consider the data $\underline{z}(\tau)$, $\tau \epsilon S$; let $\hat{x}_1$ denote the minimum-mean-square estimate of x_1, given $z(\tau)$, and $\hat{x}_2$ denote the minimum-mean-square estimate of x_2, given $z(\tau)$. Let x_1'; x_2'; $z'(\tau)$, $\tau \epsilon S$ be Gaussian random variables with the same covariance matrices as x_1, x_2 and $z(\tau)$, $\tau \epsilon S$. Then

$$\widehat{ax_1 + bx_2} = \widehat{ax_1' + bx_2'} = E\{ax_1' + bx_2' | \underline{z}(\tau), \tau \epsilon S\}$$

$$= aE\{x_1' | \underline{z}(\tau), \tau \epsilon S\} + bE\{x_2' | z(\tau) \ \tau \epsilon S\}$$

$$= a\hat{x}_1' + b\hat{x}_2' = a\hat{x}_1 + b\hat{x}_2 \quad .$$

Statement 2:— Let x be a random variable and let F be some class of functionals on the given data. Let $\hat{x}$ be the functional of this class such that

$$\mathcal{E} = E\{(x - x)^2\} \text{ is a minimum.}$$

Let f be an arbitrary functional in F. Consider the estimate $\hat{x} + \epsilon f$:

$$\mathcal{E} + \delta\mathcal{E} = E\{(x - \hat{x} - \epsilon f)^2\}$$

$$= E\{(x - \hat{x})^2\} - 2\epsilon E\{f(x - \hat{x})\} + \epsilon^2 E\{f^2\} \quad ,$$

$$\delta\mathcal{E} = -2\epsilon E\{f(x - \hat{x})\} + \epsilon^2 E\{f^2\} \quad .$$

The equation determining $\hat{x}$ is

$$\frac{\partial}{\partial\epsilon} \delta\mathcal{E}\Big|_{\epsilon=0} = -2E\{f(x - x)\} = 0 \quad . \tag{A-1}$$

To show that x is uniquely determined by Eq. (A-1) and truly yields a minimum, note that for x satisfying (A-1)

$$\delta\mathcal{E} = \epsilon^2 E\{f^2\} \gneqq 0 \quad \text{for } f \not\equiv 0 \quad .$$

Statement 3:— Define

$$Q(t) = \Phi(t)\, \Sigma(t-1)\, \Phi^T(t) + W(t-1) \quad .$$

Then Eqs. (70) and (67) combine to give

$$\Sigma(t) = Q(t) - Q(t)\, H^T(t)\, [V(t) + H(t)\, Q(t)\, H^T(t)]^{-1}\, H(t)\, Q(t) \quad .$$

Using the matrix identity*

$$(A + BD^{-1}B^T)^{-1} = A^{-1} - A^{-1}B(D + B^TA^{-1}B)^{-1}\, B^TA^{-1} \quad ,$$

---

* This identity can be derived by using the formulas for the inverse of a partitioned matrix. See Frobenius' relation in Ref. 7.

we obtain

$$\Sigma^{-1}(t) = Q^{-1}(t) + H^T(t)\, V^{-1}(t)\, H(t) \quad ,$$

from which Eq. (77) follows immediately.

Similarly, for the predictor itself, Eq. (59) can be written

$$\underline{\hat{x}}(t) = P_1(t)\, \Phi(t)\, \underline{\hat{x}}(t-1) + C(t)\, \underline{z}(t) \quad ,$$

where

$$P_1(t) = I - Q(t)\, H^T(t)\, [V(t) + H(t)\, Q(t)\, H^T(t)]^{-1}\, H(t) = \Sigma(t)\, Q^{-1}(t) \quad .$$

Now, by Eq. (70),

$$C(t)\, H(t)\, Q(t) = Q(t) - \Sigma(t) \quad ,$$

or, after manipulation,

$$\Sigma^{-1}(t)\, C(t)\, H(t) = \Sigma^{-1}(t) - Q(t)^{-1} = H^T(t)\, V^{-1}(t)\, H(t)$$

$$C(t) = \Sigma(t)\, H^T(t)\, V^{-1}(t) \quad .$$

Thus,

$$\underline{\hat{x}}(t) = \Sigma(t)\, [Q^{-1}(t)\, \Phi(t)\, \underline{\hat{x}}(t-1) + H^T(t)\, V^{-1}(t)\, \underline{z}(t)] \quad ,$$

which is Eq. (78).

REFERENCES

1. H. Cramer, Mathematical Methods of Statistics (Princeton University Press, Princeton, New Jersey, 1946), p. 300.
2. R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," J. Basic Engineering, March 1960; see also Secs. 5, 7, 11 and 12 of Ref. 5.
3. C. J. Savant, Jr., and others, Principles of Inertial Navigation (McGraw-Hill, New York, 1961).
4. J. M. Slater, J. S. Ausman, D. P. Chandler, G. A. Kachickas, R. F. Nease and R. L. Doty, "Inertial Navigation Notes" Autonetics Division, North American Aviation, Inc., Guidance Systems, Los Angeles, California (unpublished, not generally available).
5. R. E. Kalman, "New Methods and Results in Linear Prediction and Filtering Theory," Technical Report 61-1, The RIAS Corporation, Baltimore, Maryland (no date).
6. F. B. Hildebrand, Methods of Applied Mathematics (McGraw-Hill, New York, 1956), pp. 34-35, 53-59.
7. E. Bodewig, Matrix Calculus (North-Holland Publishing Company, Amsterdam, 1956).